Lec4 - ACF

Monday, 10 June 2024 15:28

Auto-Correlation Function

The auto-regressive (AR) signal model, $AR(p)$, is given by

ערכים הסטוריים

רעש

$$\hat{y}[n] = a_1 y[n-1] + a_2 y[n-2] + \cdots + a_p y[n-p] + \epsilon[n]$$

AR(1)

מודל:

$$\hat{x}[n] = a_1 x[n-1] + \epsilon[n]$$

פונ' מחיר:

$$\mathcal{L}(a_1) = \sum_n \big(\underbrace{x[n]}_{\text{ערך אמיתי}} - \underbrace{a_1 x[n-1]}_{\text{חיזוי}}\big)^2$$

$$\frac{d\mathcal{L}(a)}{da} = 2\sum_n (x[n] - a_1 x[n-1])(-x[n-1]) = 0$$

ערך אופטימלי:

$$a_1 = \frac{\sum_n x[n]x[n-1]}{\sum_n x^2[n-1]}$$

Matrix formulation

$$\underbrace{\begin{array}{c} L-1 \\ \begin{bmatrix} x[1] \\ x[2] \\ \vdots \\ x[L-2] \\ x[L-1] \end{bmatrix} \end{array}}_{\hat{\mathbf{y}}} = a_1 \underbrace{\begin{array}{c} L-1 \\ \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[L-3] \\ x[L-2] \end{bmatrix} \end{array}}_{\mathbf{x}}$$

מודל:

$$\hat{\mathbf{y}} = a_1 \mathbf{x}$$

פונ' מחיר

$$\mathcal{L} = \|\hat{\mathbf{y}} - a\mathbf{x}\|^2 = (\hat{\mathbf{y}} - a\mathbf{x})^T(\hat{\mathbf{y}} - a\mathbf{x})$$

$$a_1 = \left(\mathbf{x}^T\mathbf{x}\right)^{-1}\mathbf{x}^T\mathbf{y}$$

Least Squares (LS)

Auto-correlation function

$$\hat{x}[n] = \overset{a_1}{a_k} x[n-k] + \epsilon[n]$$

$$a_k = \frac{\sum_n x[n]x[n-k]}{\sum_n x^2[n-k]}$$

משקף של המודל

חישוב זהה

לחישוב הקודם

1) Raw auto-correlation

הפרש בזמנים

$$\underbrace{R_{\mathbf{xx}}[k]}_{\text{סימון}} = \sum_n x[n]x[n-k]$$

מצב ברירת מחדל (גם ב- Matlab)

Biased auto-correlation

2) $$R_{\mathbf{xx},biased}[k] = \frac{1}{L}\sum_n x[n]x[n-k]$$

$$= \frac{1}{L} R_{xx}[k]$$

3) Normalized auto-correlation

$$R_{\mathbf{xx},norm}[k] = \frac{R_{\mathbf{xx}}[k]}{R_{\mathbf{xx}}[0]} \lesssim a_k$$

עבור חישוב a_k

לא כולל $x^2[0]+x^2[1]$

$$\boxed{R_{\mathbf{xx}}[0] = \sum_n x^2[n]} \gtrsim \sum_n x^2[n-k] = x^2[k] + \cdots x^2[L-1]$$

4) Unbiased auto-correlation

תוספת "נרמול"

הצעה לשיפור

$$\frac{1}{L}\overbrace{\sum_n x^2[n]}^{L\ \text{מקדמים}} \approx \frac{1}{L-k}\overbrace{\sum_n x^2[n-k]}^{L-k\ \text{מקדמים}}$$

$$\frac{x^2[0] + x^2[1] + \cdots x^2[L-1]}{L} \approx \frac{x^2[k] + \cdots x^2[L-1]}{L-k}$$

$$\frac{L}{L-k}\frac{R_{\mathbf{xx}}[k]}{R_{\mathbf{xx}}[0]} = \frac{L}{L-k} R_{\mathbf{xx},norm}[k] \approx a_k$$

$$R_{\mathbf{xx},biased}[k] = \frac{1}{L-k}\sum_n x[n]x[n-k]$$

3 צורת מימוש

ב- Matlab

Python

Correlation Coefficient Interpretation

מינימום של פונ' מחיר עבור a_k אופטימלי.

$$\mathcal{L}_{min}(a_k) = \sum_{n=0}^{L-1} x^2[n] - a_k \sum_{n=0}^{L-1} x[n]x[n-k]$$

$$= R_{\mathbf{xx}}[0] - a_k R_{\mathbf{xx}}[k]$$

אחרי הצבה של a_k

$$\lesssim R_{\mathbf{xx}}[0] \boxed{\left(1 - \left(\frac{R_{\mathbf{xx}}[k]}{R_{\mathbf{xx}}[0]}\right)^2\right)}$$

שיפור בביצועים

Properties The signal energy is given by

$$E_{\mathbf{x}} = \sum_n x^2[n] = R_{\mathbf{xx}}[0]$$

and it is also the higher value of ACF,

$$R_{\mathbf{xx}}[0] \geq R_{\mathbf{xx}}[k].$$

The corresponding average power is given by

של a_k

$$\lesssim R_{\mathbf{xx}}[0]\boxed{\left(1-\left(\frac{R_{\mathbf{xx}}[k]}{R_{\mathbf{xx}}[0]}\right)^2\right)}$$ → שיפור ביצועים

$$= R_{\mathbf{xx}}[0]\left(1-R^2_{\mathbf{xx},norm}[k]\right)$$ ← $\rho_{xx}[k]$

The value of $\rho_{\mathbf{xx}}[k]$ is termed *correlation coefficient* between $x[n]$ and $x[n-k]$,

unbiased

$$\rho_{\mathbf{xx}}[k] \approx \frac{L}{L-k} R_{\mathbf{xx},norm}[k] \approx R_{\mathbf{xx},norm}[k]. \quad (20.23)$$

$$|\rho_{\mathbf{xx}}[k]| \le 1$$

$\rho = \pm 1$

מקסימום של פונ' מחיר

$$0 \le \mathcal{L}_{min}(a_k) \le \sum_{n=0}^{L-1} x^2[n] = R_{\mathbf{xx}}[0]$$

MSE and RMSE For example, the corresponding MSE and RMSE metrics are given by

מדדים לביצועים

פונ' מחיר

$$MSE(a_k) = \frac{1}{L}\mathcal{L}_{min}(a_k) \quad (20.27a)$$

$$RMSE(a_k) = \sqrt{\frac{1}{L}\mathcal{L}_{min}(a_k)} \quad (20.27b)$$

Correlation time One of the way to quantify the time ahead of the linear predictability, the correlation time is used. The correlation time is defined by the smallest time, k_c, that satisfies,

$$\rho_{\mathbf{xx}}[k_c] = 0.5 \text{ or } 0.1 \text{ or } \exp(-1) \approx 0.36 \quad (20.28)$$

The decision threshold depends on the field of application. Practically, $\rho_{\mathbf{xx}}[k > k_c] \approx 0$ is assumed.

מעבר להפרש זמן k (כמעט) אין קשר ליניארי.
= צניח

$$R_{\mathbf{xx}}[0] \ge R_{\mathbf{xx}}[k].$$

The corresponding average power is given by

$$P_{\mathbf{x}} = \frac{1}{L}\sum_n x^2[n] = R_{\mathbf{xx},biased}[0]$$

The ACF has inherent time symmetry,

$$R_{\mathbf{xx}}[k] = R_{\mathbf{xx}}[-k]$$

$k = -(L-1), \dots, 0, 1, L-1$

אין חפיפה

20.1.3 Auto-covariance

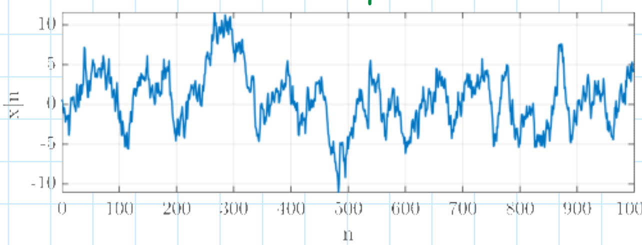
For simplicity, a zero-average, $\bar{x}[n] = 0$, was assumed. When the signals is non-zero mean, the subtraction of signal average from the signal, $x[n] = x[n] - \bar{x}[n]$ before auto-correlation calculation is termed as auto-covariance.

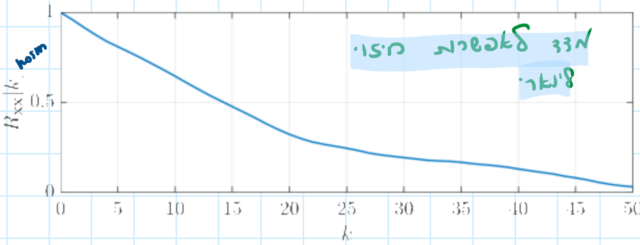
$\bar{x}[n] \ne 0$

$$\bar{x}[n] = \sum_{n=0}^{L-1} x[n] \ne 0$$

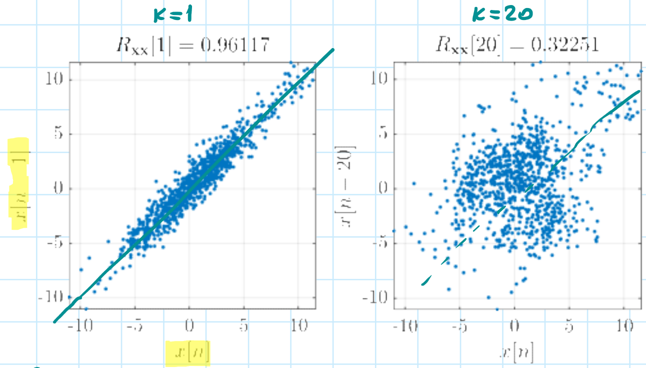
Matlab

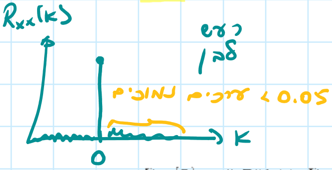
דוגמה של אות אקראי

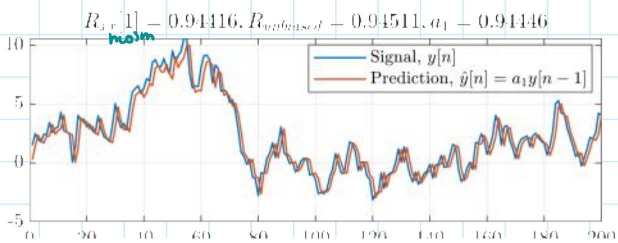


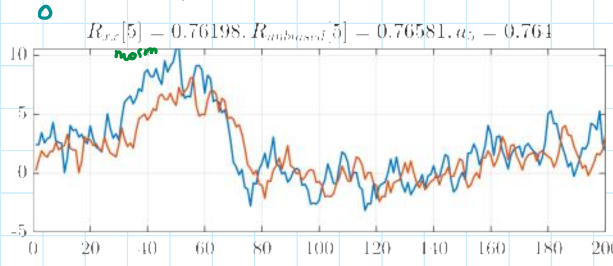


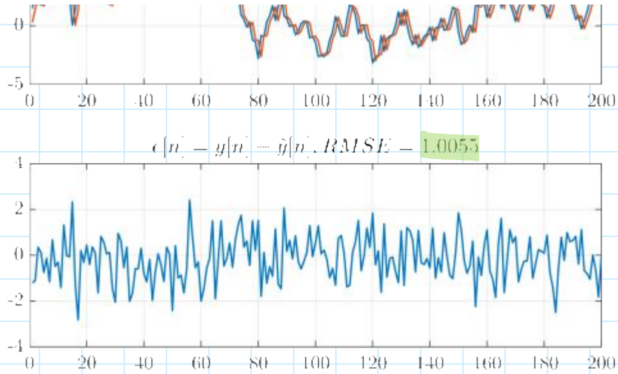
[R,lags] = xcorr(y,50,'coeff');

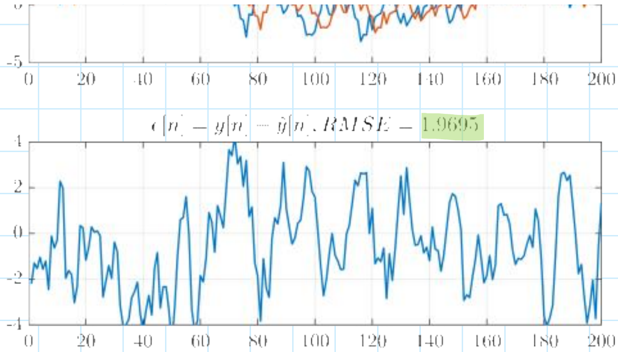












תרגיל בית

$$\mathcal{L}_{min}(a_k) = \sum_{n=0}^{L-1} x^2[n] - a_k \sum_{n=0}^{L-1} x[n]x[n-k]$$

2) חישוב תאורטי "מדויק"

$$= R_{\mathbf{xx}}[0] - a_k R_{\mathbf{xx}}[k]$$

$$\lesssim R_{\mathbf{xx}}[0]\left(1 - \left(\frac{R_{\mathbf{xx}}[k]}{R_{\mathbf{xx}}[0]}\right)^2\right)$$

3) חישוב המקורב

$$= R_{\mathbf{xx}}[0]\left(1 - R^2_{\mathbf{xx},norm}[k]\right)$$

1) חישוב מספרי ישיר

$$\mathcal{L}_{min}(a_k) = \mathbf{e}^T\mathbf{e}$$