המחלקה להנדסת חשמל ואלקטרוניקה

תאריך הבחינה: 10.03.2025
שעות הבחינה: 13:30-16:30

מבוא לאותות אקראיים

מועד ב'

ד"ר דימה בחובסקי, מר טל פאר

תשפ"ה סמסטר א'

חומר עזר - דף נוסחאות אישי (עמוד אחד), מחשבון

הוראות מיוחדות:

- השאלון כולל שאלות ללא בחירה, סך הכל של 120 נקודות.
- סעיפים הם בעלי ניקוד זהה, אלא אם צוין אחרת.
- יש לציין באופן מלא וברור את שלבי הפתרון. תשובה ללא הסבר לא תתקבלנה.
- במקום בו נדרש חישוב מספרי, יש קודם לרשום את הנוסחא, ורק אח"כ להציב!
- יש לציין יחידות למספרים, ובמידה וקיימות!
- כל השרטוטים יהיו גדולים, ברורים, עם סימון צירים!
- אין חובה להגיע לערך מספרי של הפונקציה $Q(x)$, במידה ומופיעה בתשובה.

השאלון כולל 13 דפים (כולל דף זה)

בהצלחה !

1 חציון (20 נק')

נתונה פונקציית PDF של משתנה אקראי X, מהצורה

$$f_X\left(x\right) = \frac{1}{2}e^{2x}; \quad -\infty < x < \ln(2)$$

יש למצוא:

(א) $F_X(x)$

(ב) ערך חציון של ההתפלגות.

2 חיזוי ומשתנים גאוסיים (45 נק')

נתונים משתנים אקראיים גאוסיים, $X_1 \sim N(0,1), X_2 \sim N(1,2), X_3 \sim N(0,1)$ (סימון $N(\mu, \sigma^2)$).
ידוע, שמקדם קורלציה בין X_1, X_2 הוא $\rho_{X_1X_2} = \frac{1}{\sqrt{2}}$. בנוסף,

$$Y_1 = X_2 + X_3 \quad \sim N(1,5)$$
$$Y_2 = -X_1 + 2X_3 \sim N(0,3)$$
$$Y_3 = X_1 + 2X_2$$
$$Y_4 = 3X_1 + 2X_2$$

(א) מהו חיזוי אופטימלי במובן שגיאה ריבועית מינימלית של X_2 מתוך X_1?

(ב) מהו חיזוי אופטימלי במובן שגיאה ריבועית מינימלית של Y_3 מתוך Y_4?

(ג) לכל אחד מהזוגות משתנים, **יש להוכיח**, האם הם בלתי תלויים?

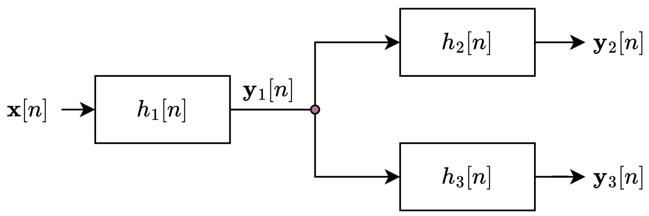
X_1, X_3 (1)

X_2, X_3 (2)

רמז: יש להיעזר בהגדרות של Y_1, Y_2.

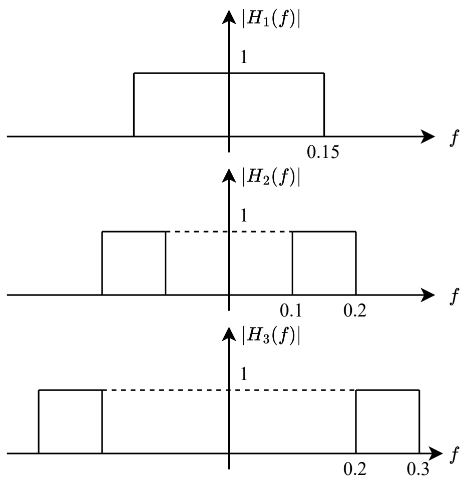
3 אותות גאוסיים ומערכות (25 נק')

נתונות 3 מערכות $h_1[n], h_2[n], h_3[n]$ בעלי כניסה $\mathbf{x}[n]$ ומוצא $\mathbf{y}_1[n], \mathbf{y}_2[n], \mathbf{y}_3[n]$, כמתואר באיור.



האות הכניסה הוא רעש לבן גאוסי בעל $R_{\mathbf{x}}[k] = 4\delta[k]$. תגובות תדר של המערכות (איור 1 הוא ערך מוחלט),

$$H_1(f) = \begin{cases} e^{-j2\pi f} & |f| \leqslant 0.15 \\ 0 & \text{אחרת} \end{cases},\ H_2(f) = \begin{cases} e^{-j2\pi f} & 0.1 \leqslant |f| \leqslant 0.2 \\ 0 & \text{אחרת} \end{cases},\ H_3(f) = \begin{cases} e^{-j2\pi f} & 0.2 \leqslant |f| \leqslant 0.3 \\ 0 & \text{אחרת} \end{cases}$$



איור 1: תגובות אמפליטודה של המערכות.

שאלה: ידוע, שלאותות $\mathbf{y}_2[n], \mathbf{y}_3[n]$ יש התפלגות גאוסית משותפת מהצורה

$$\begin{bmatrix} \mathbf{y}_2[n] \\ \mathbf{y}_3[n] \end{bmatrix} \sim N\left(\begin{bmatrix} E\left[\mathbf{y}_2[n]\right] \\ E\left[\mathbf{y}_3[n]\right] \end{bmatrix}, \begin{bmatrix} C_{\mathbf{y}_2}[0] & C_{\mathbf{y}_{23}}[0] \\ C_{\mathbf{y}_{23}}[0] & C_{\mathbf{y}_3}[0] \end{bmatrix} \right)$$

מהם הפרמטרים של ההתפלגות?

4 תלות (30 נק')

$\Theta \sim U[-\pi,\pi]$. נגדיר,

$$X = \cos(\Theta) \tag{3}$$

$$Y = \sin(\Theta) \tag{4}$$

(א) יש להוכיח שמשתנים X,Y הם חסרי קורלציה.

(ב) יש לחשב $E\left[X^2\right], E\left[Y^2\right]$.

(ג) יש להראות ע"י דוגמה, שמודבר במשתנים שהם לא בלתי תלויים. רמז: ניתן להיעזר במשוואה

$$E\left[g_1(X)g_2(Y)\right] = E\left[g_1(X)\right] E\left[g_2(Y)\right]$$

שמתקיימת עבור משתנים בלתי תלויים בלבד.

Random Processes – Formulas

1 Distributions

1.1 Continuous

	Notation	PDF	CDF	$E[X]$	Var$[X]$
Uniform	U[a,b]	$\begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b < x \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal	$N(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$	$\Phi(x)$	μ	σ^2
Exponential	$Exp(\lambda)$	$\lambda \exp(-\lambda x), x \geq 0$	$1 - \exp(-\lambda x)$	$1/\lambda$	$1/\lambda^2$

1.1.1 Q-function

Given $Y \sim N(\mu, \sigma^2)$

$$\frac{Y-\mu}{\sigma} \sim N(0,1) \tag{1}$$

$$p(Y > y) = Q\left(\frac{y-\mu}{\sigma}\right) \tag{2}$$

$$Q(x) = 1 - \Phi(x) \tag{3}$$

$$Q(-x) = 1 - Q(x) \tag{4}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{s^2}{2}\right) ds \tag{5}$$

1.2 Discrete

	Notation	PDF	CDF	$E[X]$	Var$[X]$
Bernoulli	Ber(p)	$\begin{cases} 1-p & k=0 \\ p & k=1 \end{cases}$	$\begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$	p	$p(1-p)$
Binomial	Bin(n,p)	$\binom{n}{k} p^k (1-p)^{n-k}$		np	$np(1-p)$
Geometric	Geo(p)	$p(1-p)^{k-1}$	$1-(1-p)^k$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

2 Random Variables

Definitions:

$$F_X(x) = p(X \leq x) \tag{6}$$

$$f_X(x) = \frac{\partial F_X(x)}{\partial x} \geq 0 \tag{7}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(p)\, dp \tag{8}$$

$$p(a < X \leqslant b) = F_X(b) - F_X(a) \tag{9}$$

$$f_X(x) \geq 0 \tag{10}$$

$$\int_{-\infty}^{\infty} f_X(x) dx = 1 \tag{11}$$

$$p_X[x_k] = \Pr[X = x_k] \tag{12}$$

$$0 \leq p_X[x_i] \leq 1 \ \forall i \tag{13}$$

$$\sum_i p_X[x_i] = 1 \tag{14}$$

$$F_X(x) = \Pr(X \leq x),\ x \in \mathbb{R} \tag{15}$$

$$F_X(x) = \sum_{k:x_k \leqslant x} p_X[x_k] \tag{16}$$

Expectation:

$$E[X] = \begin{cases} \int_{-\infty}^{\infty} x f_X(x) dx \\ \sum_i x_i p_X[x_i] \end{cases} \tag{17a}$$

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x) dx \\ \sum_i g(x_i) p_X[x_i] \end{cases} \tag{17b}$$

$$E[aX+b] = aE[x] + b \tag{17c}$$

Variance:

$$\begin{aligned} \mathrm{Var}[X] =& E\Big[(X - E[X])^2\Big] \\ =& E[X^2] - E^2[X] \end{aligned} \tag{18a}$$

$$\mathrm{Var}[aX+b] = a^2 \mathrm{Var}[X] \tag{18b}$$

$$\mathrm{Var}[b] = 0 \tag{18c}$$

Median: Value of m, such that $\Pr(X \leq m) \geq \frac{1}{2}$ and $\Pr(X \geq m) \geq \frac{1}{2}$

2.1 Numerical calculation

$$E[X] = \frac{1}{N} \sum_{i=1}^{N} x_i \tag{19}$$

$$\mathrm{Var}[X] = \frac{1}{N} \sum_{i=1}^{N} \big(x_i - E[X]\big)^2 \tag{20}$$

2.2 Histogram

$$p_X[x_i] \approx \frac{n_i}{N} \quad i = 1, \ldots, k \tag{21}$$

$$f_X(x_i) \approx \frac{n_i}{N} \cdot \frac{1}{\Delta x} \quad i = 1, \ldots, k \tag{22}$$

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3 Two Random Variables

3.1 Joint Distributions

Definitions:

$$F_{XY}(x,y) = p(X \leq x, Y \leq y) \tag{23a}$$

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \geqslant 0 \tag{23b}$$

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(s,p)\, dp\, ds \tag{23c}$$

$$p[x_j, y_k] = p(X = x_j, Y = y_k) \tag{24a}$$

$$F_{XY}(x,y) = p(X \leqslant x_j, Y \leqslant y_k) \tag{24b}$$

Expectation:

$$E[g(X,Y)] = \begin{cases} \iint g(x,y) f_{XY}(x,y) dx dy \\ \sum_i \sum_k g(x_i, y_k) p_{XY}[x_i, y_k] \end{cases} \tag{25a}$$

$$E[aX + bY] = aE[X] + bE[Y] \tag{25b}$$

For **independent** random variables:

$$f_{XY}(x,y) = f_X(x) f_Y(y) \tag{26a}$$

$$p_{XY}[x_k, y_j] = p_X[x_k] p_Y[y_j] \tag{26b}$$

$$F_{XY}(x,y) = F_X(x) F_Y(y) \tag{26c}$$

(26d)

Independent random variables properties:

$$E[XY] = E[X]E[Y] \tag{27a}$$

$$E[g_1(X) g_2(Y)] = E[g_1(X)] E[g_2(Y)] \tag{27b}$$

$$\mathrm{Var}[aX + bY] = a^2 \mathrm{Var}[X] + b^2 \mathrm{Var}[Y] \tag{27c}$$

Marginal distribution:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) dy \tag{28a}$$

$$p_X[x_k] = \sum_j p_{XY}[x_k, y_j] \tag{28b}$$

$$F_X(x) = F_{XY}(x, \infty) \tag{28c}$$

$$F_Y(y) = F_{XY}(\infty, y) \tag{28d}$$

3.2 Correlation, Covariance & Correlation Coefficient

- For two jointly-distributed random variables X and Y, covariance is given by

$$\begin{aligned} \mathrm{Cov}[X,Y] =& E\Big[(X - E[X])(Y - E[Y])\Big] \\ =& E[XY] - E[X]E[Y] \end{aligned} \tag{29}$$

Main covariance properties are:

$$\mathrm{Cov}[X,X] = Var[X] \tag{30a}$$

$$\mathrm{Cov}[X,Y] = \mathrm{Cov}[Y,X] \tag{30b}$$

$$\mathrm{Cov}[X,a] = 0 \tag{30c}$$

$$\mathrm{Cov}[aX, bY] = ab\,\mathrm{Cov}[X, Y] \tag{30d}$$
$$\mathrm{Cov}[X, Y] = \mathrm{Cov}[X + a, Y + b] \tag{30e}$$
$$\mathrm{Var}[aX \pm bY] = a^2\mathrm{Var}[X] + b^2\mathrm{Var}[Y] \pm 2ab\,\mathrm{Cov}[X, Y] \tag{30f}$$

- Correlation coefficient (also termed as Pearson product-moment correlation coefficient) is given by

$$\rho_{XY} = \frac{\mathrm{Cov}[X, Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} \tag{31}$$

such that $|\rho_{XY}| \leq 1$.

3.3 MMSE Linear Prediction

Mean square error (MSE) of predictor $\hat{Y}$ is given by

$$mse = E[(Y - \hat{Y})^2] \tag{32}$$

Linear prediction of $\hat{Y} = ax + b$ for $X = x$ is

$$\hat{Y} = E[Y] + \frac{\mathrm{Cov}[X, Y]}{\mathrm{Var}[X]}\left(x - E[X]\right) \tag{33}$$

and

$$mse_{min} = E\left[\left(Y - (aX + b)\right)^2\right] = \mathrm{Var}[Y]\,(1 - \rho_{XY}^2) \tag{34}$$

When X, Y are jointly Gaussian, this prediction is optimal among **all** possible predictors

3.4 Relations

- When X and Y are *orthogonal*, $E[XY] = 0$.
- When X and Y are *uncorrelated*, $\mathrm{Cov}[X, Y] = \rho_{XY} = 0$.
- When X and Y are *independent*, they are also uncorrelated (see also Eqs. 26).
- When X and Y are *jointly* Gaussian and uncorrelated $\Rightarrow$ X and Y are independent.

3.5 Bi-variate Normal Distribution

Joint Gaussian distribution of X_1 and X_2

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \mathrm{C}_{\mathbf{X}}\right) \tag{35}$$

with covariance matrix

$$\mathrm{C}_{\mathbf{X}} = \begin{bmatrix} \mathrm{Cov}[X_1, X_1] & \mathrm{Cov}[X_1, X_2] \\ \mathrm{Cov}[X_2, X_1] & \mathrm{Cov}[X_2, X_2] \end{bmatrix} \tag{36}$$

Important properties:

- Sum of independent Gaussian variables is a Gaussian variable.
- Random vector $[X_1, \ldots, X_n]$ is **jointly** Gaussian distributed, iff (if and only if) for all possible real vectors $\mathbf{a} = (a_1, \ldots, a_n)^T$ linear combination $Y = a_1X_1 + \cdot + a_nX_n$ is Gaussian distributed.
- If jointly distributed Gaussian random variables are *uncorrelated*, they are also *independent*

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4 Random Processes – General Properties

- PDF & CDF

$$F_{\mathbf{x}}(x; t) = p(\mathbf{x}(t) \leq x) \tag{37a}$$
$$f_{\mathbf{x}}(x; t) = \frac{\partial}{\partial x}F_{\mathbf{x}}(x; t) \tag{37b}$$
$$p_{\mathbf{x}}[x_k; n] = p(\mathbf{x}[n] = x_k) \tag{37c}$$

- Average:

$$E[\mathbf{x}(t)] = \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x; t)dx \tag{38a}$$
$$E\left[\mathbf{x}[n]\right] = \sum_i x_i p_{\mathbf{x}}[x_k; n] \tag{38b}$$

- Variance:

$$\mathrm{Var}\left[\mathbf{x}(t)\right] = E\left[\mathbf{x}^2(t)\right] - E^2\left[\mathbf{x}(t)\right] = \sigma_{\mathbf{x}}^2(t) \tag{39a}$$
$$\mathrm{Var}\left[\mathbf{x}[n]\right] = E\left[\mathbf{x}^2[n]\right] - E^2\left[\mathbf{x}[n]\right] = \sigma_{\mathbf{x}}^2[n] \tag{39b}$$

- Auto-correlation

$$R_{\mathbf{x}}(t_1, t_2) = E[\mathbf{x}(t_1)\mathbf{x}(t_2)] \tag{40a}$$
$$R_{\mathbf{x}}(t, t + \tau) = E[\mathbf{x}(t)\mathbf{x}(t + \tau)] \tag{40b}$$
$$R_{\mathbf{x}}(t_1, t_2) = R_{\mathbf{x}}(t_2, t_1) \tag{40c}$$

$$R_{\mathbf{x}}(t,t) = E[\mathbf{x}^2(t)] \tag{40d}$$
$$R_{\mathbf{x}}[n_1,n_2] = E\left[\mathbf{x}[n_1]\mathbf{x}[n_2]\right] \tag{40e}$$
$$R_{\mathbf{x}}[n,n] = E\left[\mathbf{x}^2[n]\right] \tag{40f}$$

- Auto-covariance

$$C_{\mathbf{x}}(t_1,t_2) = E\left[\left\{\mathbf{x}(t_1) - E[\mathbf{x}(t_1)]\right\}\left\{\mathbf{x}(t_2) - E[\mathbf{x}(t_2)]\right\}\right] \tag{41}$$
$$= R_{\mathbf{x}}(t_1,t_2) - E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)] \tag{42}$$
$$C_{\mathbf{x}}[n_1,n_2] = E\left[\left\{\mathbf{x}[n_1] - E[\mathbf{x}[n_1]]\right\}\left\{\mathbf{x}[n_2] - E[\mathbf{x}[n_2]]\right\}\right] \tag{43}$$
$$= R_{\mathbf{x}}[n_1,n_2] - E\left[\mathbf{x}[n_1]\right]E\left[\mathbf{x}[n_2]\right] \tag{44}$$

$$C_{\mathbf{x}}(t,t) = \mathrm{Var}\left[\mathbf{x}(t)\right] \tag{45a}$$
$$C_{\mathbf{x}}[n,n] = \mathrm{Var}\left[\mathbf{x}[n]\right] \tag{45b}$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(t_1,t_2) = \frac{C_{\mathbf{x}}(t_1,t_2)}{\sqrt{C_{\mathbf{x}}(t_1,t_1)C_{\mathbf{x}}(t_2,t_2)}} \tag{46a}$$
$$\left|\rho_{\mathbf{x}}(t_1,t_2)\right| \leqslant 1 \tag{46b}$$

  - When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *orthogonal*, $R_{\mathbf{x}}(t_1,t_2) = 0$.
  - When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *uncorrelated*, $C_{\mathbf{x}}(t_1,t_2) = \rho_{\mathbf{x}}(t_1,t_2) = 0$.
  - When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *independent*, $R_{\mathbf{x}}(t_1,t_2) = E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)]$.

- Power

$$P_{\mathbf{x}}(t) = E\left[\mathbf{x}^2(t)\right] \tag{47a}$$
$$P_{\mathbf{x}}[n] = E\left[\mathbf{x}^2[n]\right] \tag{47b}$$

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5 Wide-Sense Stationary (WSS) Process

Definition:

$$E[\mathbf{x}(t)] = E[\mathbf{x}(0)] = \mu_{\mathbf{x}} = \text{const} \tag{48a}$$
$$R_{\mathbf{x}}(t_1,t_2) = R_{\mathbf{x}}(\tau = |t_2 - t_1|), \quad \forall t_1, t_2 \tag{48b}$$
$$E[\mathbf{x}[n]] = E[\mathbf{x}[0]] = \mu_{\mathbf{x}} = \text{const} \tag{48c}$$
$$R_{\mathbf{x}}[n_1,n_2] = R_{\mathbf{x}}(k = |n_2 - n_1|), \quad \forall n_1, n_2 \tag{48d}$$

- Auto-correlation

$$R_{\mathbf{x}}(\tau) = E[\mathbf{x}(t)\mathbf{x}(t+\tau)] \tag{49a}$$
$$R_{\mathbf{x}}[k] = E\left[\mathbf{x}[n]\mathbf{x}[n+k]\right] \tag{49b}$$

  Properties:

$$R_{\mathbf{x}}(-\tau) = R_{\mathbf{x}}(\tau) \tag{50a}$$
$$R_{\mathbf{x}}(0) = E[\left|\mathbf{x}(0)\right|^2] = E[\left|\mathbf{x}(t)\right|^2] \tag{50b}$$
$$\mathrm{Var}\left[\mathbf{x}(t)\right] = C_{\mathbf{x}}(0) = \sigma_{\mathbf{x}}^2 \tag{50c}$$
$$R_{\mathbf{x}}(0) \geqslant \left|R_{\mathbf{x}}(\tau)\right| \tag{50d}$$

- Auto-covariance

$$C_{\mathbf{x}}(\tau) = R_{\mathbf{x}}(\tau) - \mu_{\mathbf{x}}^2 \tag{51a}$$
$$C_{\mathbf{x}}[k] = R_{\mathbf{x}}[k] - \mu_{\mathbf{x}}^2 \tag{51b}$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(\tau) = \frac{C_{\mathbf{x}}(\tau)}{C_{\mathbf{x}}(0)} \tag{52a}$$
$$\rho_{\mathbf{x}}[k] = \frac{C_{\mathbf{x}}[k]}{C_{\mathbf{x}}[0]} \tag{52b}$$

5.1 Power Spectral Density (PSD)

$$S_{\mathbf{x}}(F) = \mathcal{F}\left\{R_{\mathbf{x}}(\tau)\right\} = \qquad -\infty \leq F \leq \infty$$
$$= \int_{-\infty}^{\infty} R_{\mathbf{x}}(\tau)\exp\left(-j2\pi F\tau\right)d\tau \tag{53a}$$
$$R_{\mathbf{x}}(\tau) = \mathcal{F}^{-1}\left\{S_{\mathbf{x}}(F)\right\} =$$
$$= \int_{-\infty}^{\infty} S_{\mathbf{x}}(f)\exp\left(j2\pi F\tau\right)dF \tag{53b}$$
$$S_{\mathbf{x}}(f) = \mathrm{DTFT}\left\{R_{\mathbf{x}}[k]\right\} = \sum_{k=-\infty}^{\infty} R_{\mathbf{x}}[k]e^{-j2\pi fk} \tag{53c}$$

Properties:

$$S_{\mathbf{x}}(F) = S_{\mathbf{x}}(-F) \tag{54a}$$
$$S_{\mathbf{x}}(F) \geqslant 0,\ \forall F \tag{54b}$$
$$S_{\mathbf{x}}(F) \in \mathbb{R} \tag{54c}$$
$$S_{\mathbf{x}}(f) = S_{\mathbf{x}}(-f) \tag{54d}$$
$$S_{\mathbf{x}}(f) \geqslant 0,\ \forall f \tag{54e}$$
$$S_{\mathbf{x}}(f) \in \mathbb{R} \tag{54f}$$
$$S_{\mathbf{x}}(f) = S_{\mathbf{x}}(f+1) \tag{54g}$$

Average power

$$P_{\mathbf{x}} = E\left[\mathbf{x}^2(t)\right] = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(F)dF \tag{55a}$$

$$P_{\mathbf{x}} = E\left[\mathbf{x}^2[n]\right] = R_{\mathbf{x}}[0] = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{\mathbf{x}}(f)df \tag{55b}$$

5.2 White Noise & White Gaussian Noise (WGN) Process

White noise process is SSS (WSS) process that is characterized by

$$R_{\mathbf{n}}(\tau) = \sigma^2\delta(\tau) \tag{56a}$$
$$S_{\mathbf{n}}(F) = \sigma^2 \quad \forall F \tag{56b}$$

For WGN process, $\mathbf{n}(t) \sim N(0, \sigma^2)$,

$$R_{\mathbf{n}}(\tau) = \frac{N_0}{2}\delta(\tau) \tag{57a}$$
$$S_{\mathbf{n}}(F) = \frac{N_0}{2} \quad \forall F \tag{57b}$$

5.3 Relation Between Covariance Matrix & Auto-covariance

Given WSS process $\mathbf{x}(t)$, the corresponding correlation matrix of $\mathbf{X} = [\mathbf{x}(t_1), \ldots, \mathbf{x}(t_N)]^T$ is given by

$$R_{\mathbf{X}} = E\left[\mathbf{X}\mathbf{X}^T\right] \tag{58}$$
$$R_{\mathbf{X}}(i,j) = E\left[X_i X_j\right] = R_{\mathbf{x}}\left(|t_i - t_j|\right) \tag{59}$$

6 Cross-Signal

- Cross-correlation

$$R_{\mathbf{xy}}(t_1, t_2) = E[\mathbf{x}(t_1)\mathbf{y}(t_2)] \tag{60}$$

- Cross-covariance

$$C_{\mathbf{xy}}(t_1, t_2) = R_{\mathbf{xy}}(t_1, t_2) - E[\mathbf{x}(t_1)]E[\mathbf{y}(t_2)] \tag{61}$$

- Correlation Coefficient

$$\rho_{\mathbf{xy}}(t_1, t_2) = \frac{C_{\mathbf{xy}}(t_1, t_2)}{\sqrt{C_{\mathbf{x}}(t_1, t_1)C_{\mathbf{y}}(t_2, t_2)}} \tag{62}$$

6.1 WSS Cross-signal

- $\mathbf{x}(t), \mathbf{y}(t)$ are jointly WSS, if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ each of them is WSS and

$$R_{\mathbf{xy}}(\tau) = E[\mathbf{x}(t)\mathbf{y}(t+\tau)] \tag{63}$$

- When $\mathbf{x}(t)$ and $\mathbf{y}(t+\tau)$ are *uncorrelated jointly WSS*, $C_{\mathbf{xy}}(\tau) = 0$.

  Properties

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{yx}}(-\tau) \tag{64a}$$
$$\left|R_{\mathbf{xy}}(\tau)\right| \leqslant \sqrt{R_{\mathbf{x}}(0)R_{\mathbf{y}}(0)} \tag{64b}$$
$$\left|R_{\mathbf{xy}}(\tau)\right| \leqslant \frac{1}{2}\left[R_{\mathbf{x}}(0) + R_{\mathbf{y}}(0)\right] \tag{64c}$$

- Cross-covariance

$$C_{\mathbf{xy}}(\tau) = R_{\mathbf{xy}}(\tau) - \mu_{\mathbf{x}}\mu_{\mathbf{y}} \tag{65}$$

- Cross-PSD

$$S_{\mathbf{xy}}(f) = \mathcal{F}\left\{R_{\mathbf{xy}}(\tau)\right\} \tag{66}$$

  Properties

$$S_{\mathbf{xy}}(f) = S_{\mathbf{yx}}(-f) = S^*_{\mathbf{xy}}(-f) \tag{67}$$

  Correlation coefficient

$$\rho_{\mathbf{xy}}(\tau) = \frac{C_{\mathbf{xy}}(\tau)}{C_{\mathbf{xy}}(0)} \tag{68}$$

- Coherence

$$\gamma_{\mathbf{xy}}(f) = \frac{S_{\mathbf{xy}}(f)}{\sqrt{S_{\mathbf{x}}(f)S_{\mathbf{y}}(f)}} \tag{69}$$

7 LTI and WSS Random Process

Output of LTI system with impulse response $h(t)$ and random process $x(t)$,

$$y(t) = x(t) * h(t) \tag{70}$$

Average

$$m_{\mathbf{y}} = m_{\mathbf{x}} \int_{-\infty}^{\infty} h(s)ds = m_{\mathbf{x}}H(F=0) \tag{71}$$

Cross-correlation & cross-covariance:

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) \tag{72a}$$
$$C_{\mathbf{xy}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) \tag{72b}$$
$$R_{\mathbf{yx}}(\tau) = R_{\mathbf{x}}(\tau) * h(-\tau) \tag{72c}$$
$$C_{\mathbf{yx}}(\tau) = C_{\mathbf{x}}(\tau) * h(-\tau) \tag{72d}$$
$$R_{\mathbf{y}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \tag{72e}$$

$$C_{\mathbf{y}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \tag{72f}$$

Power-Spectral Density (PSD) & Cross-PSD: Given frequency response
$H(F) = \mathcal{F}\{h(\tau)\}, \; H^*(F) = \mathcal{F}\{h(-\tau)\}$

$$S_{\mathbf{xy}}(F) = S_{\mathbf{x}}(F) H(F) \tag{73a}$$
$$S_{\mathbf{yx}}(F) = S_{\mathbf{x}}(F) H^*(F) \tag{73b}$$
$$S_{\mathbf{y}}(F) = S_{\mathbf{x}}(F) \left|H(F)\right|^2 \tag{73c}$$

Power of the process:

$$P_y = R_{\mathbf{y}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(F) \left|H(F)\right|^2 dF \tag{74}$$

Same process passes two different systems

$$R_{\mathbf{yz}}(\tau) = R_{\mathbf{x}}(\tau) * h_1(-\tau) * h_2(\tau) \tag{75}$$
$$S_{\mathbf{yz}}(F) = S_{\mathbf{x}}(F) H_1^*(F) H_2(F) \tag{76}$$

7.1 Discrete-Time

Auto-correlation

$$H(z) = \mathscr{Z}\{h[n]\} = \frac{B(z)}{A(z)}$$
$$\mathscr{Z}\{h[n] * h[-n]\} = \frac{B(z)B(z^{-1})}{A(z)A(z^{-1})}$$
$$S_{\mathbf{x}}(z) = \mathscr{Z}\{R_{\mathbf{x}}[n]\}$$
$$h[k] * h[-k] = \sum_m h[m]h[m+k]$$

PSD

$$S_{\mathbf{xy}}(z) = S_{\mathbf{x}}(z)H(z) \tag{77a}$$
$$S_{\mathbf{yx}}(z) = S_{\mathbf{x}}(z)H(z^{-1}) \tag{77b}$$
$$S_{\mathbf{y}}(z) = S_{\mathbf{x}}(z)H(z)H(z^{-1}) \tag{77c}$$

Two different systems

$$R_{\mathbf{yz}}[k] = R_{\mathbf{x}}[k] * h_1[-k] * h_2[k] \tag{78a}$$
$$S_{\mathbf{yz}}(f) = S_{\mathbf{x}}(f)H_1^*(f)H_2(f) \tag{78b}$$
$$S_{\mathbf{yz}}(z) = S_{\mathbf{x}}(z)H_1(1/z)H_2(z) \tag{78c}$$

Power of the process:

$$P_x = R_{\mathbf{x}}[0] = \int_{-1/2}^{1/2} S_{\mathbf{x}}(f)\, df \tag{79a}$$
$$P_y = R_{\mathbf{y}}[0] = \int_{-1/2}^{1/2} S_{\mathbf{x}}(f) \left|H(f)\right|^2 df \tag{79b}$$

Average

$$\mu_{\mathbf{y}} = \mu_{\mathbf{x}} \sum_m h[m] \tag{80}$$

For white Gaussian noise input

$$\mathrm{Var}\left[\mathbf{y}[n]\right] = \mathrm{Var}\left[\mathbf{x}[n]\right] \sum_m h^2[m] \tag{81}$$

7.2 Gaussian Process

A Gaussian process $\mathbf{x}(t)$ a random process that for $\forall k > 0$ and for all times $t_1, \ldots, t_k$, the set of random variable $\mathbf{x}(t_1), \ldots, \mathbf{x}(t_k)$ is jointly Gaussian.

Properties:

- WSS Gaussian process is SSS.
- Gaussian process $\mathbf{x}(t)$ that passes through LTI system, $\mathbf{y}(t) = h(t) * \mathbf{x}(t)$, is also Gaussian process that may be described by the change of expectation and auto-correlation,

$$E[\mathbf{y}(t)] = E[\mathbf{x}(t)] \int_{-\infty}^{\infty} h(s)ds \tag{82a}$$
$$= E[\mathbf{x}(t)]H(0), \quad H(F) = \mathscr{F}\{h(t)\}$$
$$C_{\mathbf{y}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \tag{82b}$$

- The resulting autocorrelation may be used for producing the correspondent covariance matrix $C_{\mathbf{Y}}$ of a multivariate Gaussian $\mathbf{Y} = [\mathbf{y}(t_1), \ldots, \mathbf{y}(t_N)]^T$

7.3 Linear Prediction

Given N samples of process $\mathbf{x}[n]$, and predictor

$$\hat{\mathbf{x}}[n+1] = \sum_{i=1}^{N} a_i \mathbf{x}[n-i+1], \tag{83}$$

the mean-square error is given by

$$\begin{aligned} mse &= E\left[\left(\mathbf{x}[n+1] - \hat{\mathbf{x}}[n+1]\right)^2\right] \\ &= E\left[\left(\mathbf{x}[n+1] - a_0\mathbf{x}[n] - a_1\mathbf{x}[n-1] - \ldots - a_N\mathbf{x}[n-N]\right)^2\right] \end{aligned} \tag{84}$$

and the values of a_i are given by a solution of

$$\begin{bmatrix} R_{\mathbf{x}}[0] & R_{\mathbf{x}}[1] & \cdots & R_{\mathbf{x}}[N-1] \\ R_{\mathbf{x}}[1] & R_{\mathbf{x}}[0] & \cdots & R_{\mathbf{x}}[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ R_{\mathbf{x}}[N-1] & R_{\mathbf{x}}[N-2] & \cdots & R_{\mathbf{x}}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} R_{\mathbf{x}}[1] \\ R_{\mathbf{x}}[2] \\ \vdots \\ R_{\mathbf{x}}[N] \end{bmatrix} \tag{85}$$

and the resulting minimum MSE is

$$mse_{min} = R_{\mathbf{x}}[0] - \sum_{i=1}^{N} a_i R_{\mathbf{x}}[i] \tag{86}$$

7.4 Match Filter

The goal of filter $h(t)$ is to provide maximum SNR at time $t = t_0$ for *deterministic* signal $x(t)$ and noise $n(t)$.

$$H(f) = \alpha \frac{X^*(f)}{S_n(f)} e^{-j2\pi f t_0} \tag{87}$$

$$y(t) = \frac{1}{\alpha} R_v(t - t_0) \tag{88}$$

For white noise, $n(t)$, with $S_N(f) = N_0/2$, the filter is given by

$$H_{\mathrm{mf}}(f) = X^*(f) e^{-j2\pi f t_0} \longleftrightarrow h_{\mathrm{mf}}(t) = x(t_0 - t) \tag{89}$$

and the resulting maximum SNR is given by

$$SNR_{\max} = \frac{2}{N_0} \int_{-\infty}^{\infty} \left|X(f)\right|^2 df = \frac{2E_x}{N_0} \tag{90}$$

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8 Different Supplementary Formulas

8.1 Derivatives

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\frac{d}{dx} \exp\left[f(x)\right] = \exp\left[f(x)\right] \frac{d}{dx} f(x)$$

8.2 Integrals

8.2.1 Indefinite

$$\int x^n dx = \frac{1}{n+1} x^{n+1},\ n \neq -1$$

$$\int \exp(ax) dx = \frac{1}{a} \exp(ax)$$

$$\int x \exp(ax) dx = \exp(ax) \left[\frac{x}{a} - \frac{1}{a^2}\right]$$

$$\int x^2 \exp(ax) dx = \exp(ax) \left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right]$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\int \cos(ax + b) dx = \frac{\sin(ax + b)}{a}$$

8.2.2 Definite

$$\int_0^{\infty} \exp\Big(-a^2 x^2\Big) dx = \frac{\sqrt{\pi}}{2a}$$

$$\int_0^{\infty} x^2 \exp\Big(-a^2 x^2\Big) dx = \frac{\sqrt{\pi}}{4a^3}$$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x - a) dx = f(a)$$

8.3 Fourier Transform

8.3.1 Properties

$$\frac{d^n}{dt^n} g(t) \overset{\mathscr{F}}{\Longleftrightarrow} (j2\pi F)^n G(F)$$

$$g(-t) \overset{\mathscr{F}}{\Longleftrightarrow} G^*(F)$$

$$g(t - t_0) \overset{\mathscr{F}}{\Longleftrightarrow} G(F) e^{-j2\pi F t_0}$$

$$g(t) e^{j2\pi f_0 t} \overset{\mathscr{F}}{\Longleftrightarrow} G(F - F_0)$$

8.3.2 Transform pairs

$$u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2}\left(\frac{1}{j\pi F} + \delta(F)\right)$$

$$\exp(-at)u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{a + j2\pi F}$$

$$t\exp(-at)u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{(a + j2\pi F)^2}$$

$$\exp\big(-a|t|\big) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{2a}{a^2 + 4\pi^2 F^2}$$

$$\exp\Big(-at^2\Big) \overset{\mathscr{F}}{\Longleftrightarrow} \sqrt{\frac{\pi}{a}}\exp\left(-\frac{(\pi F)^2}{a}\right)$$

$$\cos(2\pi f_a t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2}\Big[\delta(F - F_a) + \delta(F + F_a)\Big]$$

$$\sin(2\pi f_a t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2j}\Big[\delta(F - F_a) - \delta(F + F_a)\Big]$$

$$u(t + a) - u(t - a) \overset{\mathscr{F}}{\Longleftrightarrow} \operatorname{sinc}(2\pi F a) \qquad \text{pulse in time}$$

$$\operatorname{sinc}(2\pi F a) \overset{\mathscr{F}}{\Longleftrightarrow} u(F + a) - u(F - a)$$

8.4 Convolution

$$x(t) * y(t) = \int_{-\infty}^{\infty} f(s)g(t - s)ds$$

$$x(t) * y(t) \overset{\mathscr{F}}{\Longleftrightarrow} X(F)Y(F)$$

$$\delta(t) * y(t) = y(t)$$

8.5 Trigonometry

$$\sin^2(\alpha) = \frac{1}{2}\big(1 - \cos(2\alpha)\big)$$

$$\cos^2(\alpha) = \frac{1}{2}\big(1 + \cos(2\alpha)\big)$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\big[\cos(\alpha + \beta) + \cos(\alpha - \beta)\big]$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\big[\cos(\alpha - \beta) - \cos(\alpha + \beta)\big]$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\Big[\sin(\alpha - \beta) + \sin\big((\alpha + \beta)\big)\Big]$$

8.6 Matrices

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$$\det[\mathbf{A}] = ad - bc$$

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

9 Discrete-Time

Series sum

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r}$$

$$\sum_{n=N_1}^{N_2-1} r^n = \frac{r^{N_1} - r^{N_2}}{1 - r} \quad N_1 \leq N_2$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r} \quad |r| < 1$$

$$\sum_{n=N_1}^{\infty} r^n = \frac{1}{1 - r^{N_1}} \quad |r| < 1$$

$$\sum_{n=0}^{\infty} nr^n = \frac{r}{(1 - r)^2} \quad |r| < 1$$

9.1 Z-transforms

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$$

Signal	Z transform	ROC
$\delta[n]$	1	$\mathbb{C}$
$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	$\mathbb{C} - \{0\}$ if $m > 0$, $\mathbb{C} - \{\infty\}$ if $m < 0$
$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > a$
$-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < a$

Property	Discrete Signal	Z transform	ROC
Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(z) + a_2X_2(z)$	includes $R_1 \cap R_2$
Time shift	$x[n-n_0]$	$z^{-n_0}X(z)$	R
Frequency scaling	$z_0^n x[n]$	$X\left(\dfrac{z}{z_0}\right)$	$\|z_0\|R$
Time reversal	$x[-n]$	$X(z^{-1})$	R^{-1} if $m < 0$
Convolution	$(x_1 * x_2)[n]$	$X_1(z)X_2(z)$	$R_1 \cap R_2$ (or possibly more)
Accumulation	$\displaystyle\sum_{k=-\infty}^{n} x[k]$	$\dfrac{X(z)}{1-z^{-1}}$	$R \cap \{\|z\| > 1\}$

9.2 DTFT

$$X(f) = X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{jn\omega}d\omega.$$