המחלקה להנדסת חשמל ואלקטרוניקה

תאריך הבחינה: 09.02.2022
שעות הבחינה: שעתיים

מבוא לאותות אקראיים

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הוראות מיוחדות:

- השאלון כולל שאלות ללא בחירה, סך הכל של 107 נקודות.
- סעיפים הם בעלי ניקוד זהה, אלא אם צוין אחרת.
- יש לציין באופן מלא וברור את שלבי הפתרון. תשובה ללא הסבר לא תתקבלנה.
- במקום בו נדרש חישוב מספרי, יש קודם לרשום את הנוסחא, ורק אח״כ להציב!
- יש לציין יחידות למספרים, ובמידה וקיימות!
- כל השרטוטים יהיו גדולים, ברורים, עם סימון צירים!
- אין חובה להגיע לערך מספרי של הפונקציה $Q(x)$, במידה ומופיעה בתשובה.

השאלון כולל 11 דפים (כולל דף זה)

בהצלחה!

1 תהליכים בזמן בדיד (91 נק')

נתונים תהליכים אקראיים:

- $\mathbf{x}_1[n]$, מתפלג גאוסית, $\mathbf{x}_1[n] \sim N(\mu_1, \sigma_1^2)$ עם $R_1[k]$.
- $\mathbf{x}_2[n]$, מתפלג גאוסית, $\mathbf{x}_2[n] \sim N(\mu_2, \sigma_2^2)$ עם $R_2[k]$.
- ניתן להניח אי תלות בין $\mathbf{x}_1[n], \mathbf{x}_2[n]$.

נתונים קשרים הבאים:

$$\mathbf{y}[n] = \mathbf{x}_1[n] + \mathbf{x}_2[n-1]$$
$$\mathbf{z}[n] = \mathbf{x}_1[n-1] + \mathbf{x}_2[n]$$

(א) יש להוכיח, ש- $\mathbf{y}[n]$ הוא WSS.

(ב) מהו הספק $P_{\mathbf{y}}$? מהי הדרך לרשום הספק תוך שימוש בפרמטרים $\mu_1, \mu_2, \sigma_1, \sigma_2$ בלבד?

(ג) מהו $C_{\mathbf{y}}[k]$?

(ד) מהי התפלגות של $\mathbf{y}[n]$?

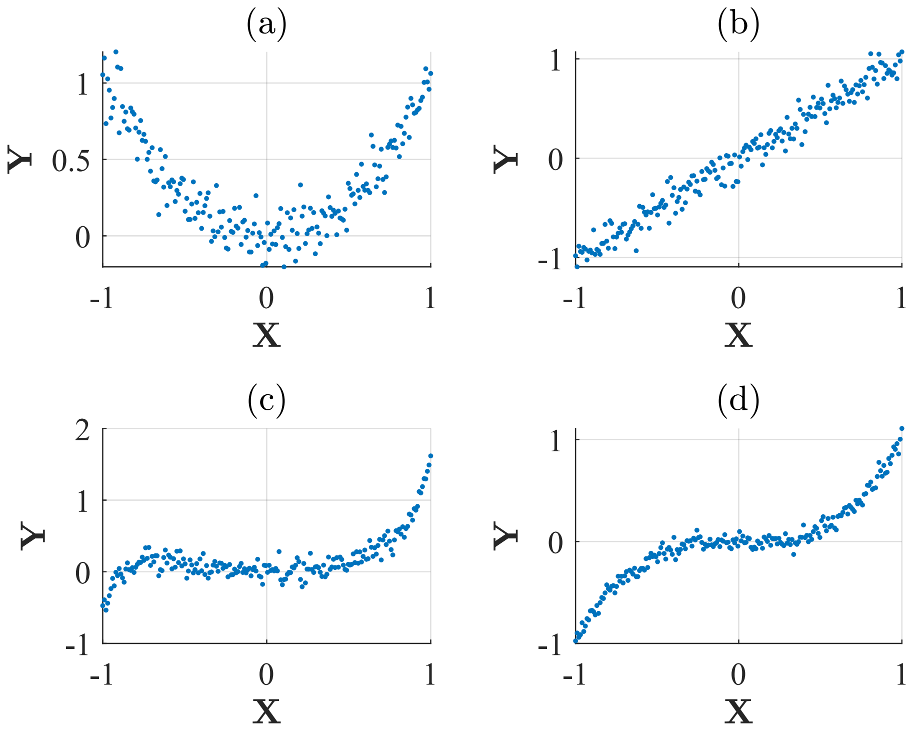
(ה) מהי התפלגות של $\mathbf{y}[0] + \mathbf{y}[1]$?

(ו) האם $\mathbf{y}[n], \mathbf{z}[n]$ הם joint-WSS?

(ז) עבור סעיף זה בלבד, $\mu_1 = \mu_2 = 0$. מהו מקדם חיזוי לינארי אופטימלי a מהצורה $\hat{z}[n] = a\mathbf{y}[n]$?

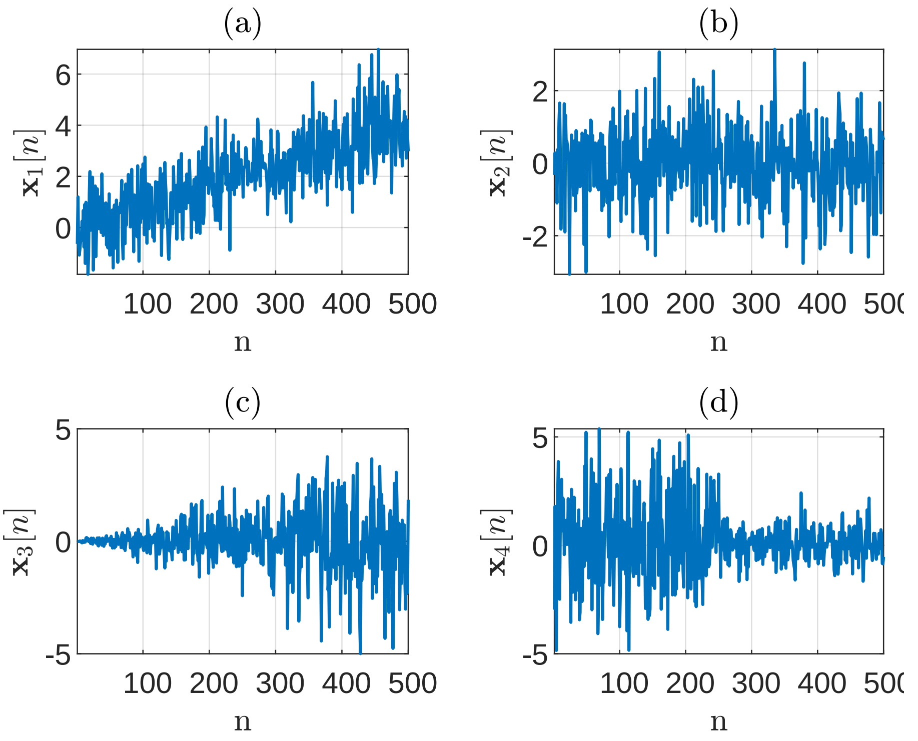
2 ניתוח גרפי (16 נק')

(א) נתונות תוצאות של 4 ניסויים דו-ממדיים שונים. יש למיין מהגבוה לנמוך את הסדר של הניסויים ע"פ ערך מוחלט של המקדם הקורלציה בין X, Y, $|\rho_{XY}|$. יש **להסביר/לנמק** את תשובתך.



איור 1 : יש לרשום את הסדר של הניסויים ע"פ ערך מוחלט של המקדם הקורלציה, מהגבוה לנמוך.

(ב) נתונות תוצאות של 4 ניסויים שונים של אותות אקראיים. עבור כל אחד מהאותות, יש לציין **ולנמק** האם מדובר באות סטציאונרי.



איור 2 : יש לציין/לנמק האם מדובר באות סטציאונרי.

Random Processes – Formulas

1 Distributions

1.1 Continuous

	Notation	PDF	CDF	$E[X]$	$\text{Var}[X]$
Uniform	U[a,b]	$\begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b < x \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal	$N(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$	$\Phi(x)$	μ	σ^2
Exponential	$Exp(\lambda)$	$\lambda \exp\left(-\lambda x\right), x \geq 0$	$1 - \exp\left(-\lambda x\right)$	$1/\lambda$	$1/\lambda^2$

1.1.1 Q-function

Given $Y \sim N(\mu, \sigma^2)$

$$\frac{Y-\mu}{\sigma} \sim N(0,1) \tag{1}$$

$$p(Y > y) = Q\left(\frac{y-\mu}{\sigma}\right) \tag{2}$$

$$Q(x) = 1 - \Phi(x) \tag{3}$$

$$Q(-x) = 1 - Q(x) \tag{4}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{s^2}{2}\right) ds. \tag{5}$$

1.2 Discrete

	Notation	PMF	CDF	$E[X]$	$\text{Var}[X]$
Bernoulli	Ber(p)	$\begin{cases} 1-p & k=0 \\ p & k=1 \end{cases}$	$\begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$	p	$p(1-p)$

2 Random Variables

Definitions:

$$F_X(x) = p(X \leq x) \tag{6}$$

$$f_X(x) = \frac{\partial F_X(x)}{\partial x} \geq 0 \tag{7}$$

$$F_X(x) = \int_{-\infty}^x f_X(p)\, dp \tag{8}$$

$$p(a < X \leqslant b) = F_X(b) - F_X(a) \tag{9}$$

$$f_X(x) \geq 0 \tag{10}$$

$$\int_{-\infty}^\infty f_X(x) dx = 1 \tag{11}$$

$$p_X[x_k] = \Pr[X = x_k] \tag{12}$$

$$0 \leq p_X[x_i] \leq 1 \ \forall i \tag{13}$$

$$\sum_i p_X[x_i] = 1 \tag{14}$$

$$F_X(x) = \Pr(X \leq x), \ x \in \mathbb{R} \tag{15}$$

$$F_X(x) = \sum_{k: x_k \leqslant x} p_X[x_k] \tag{16}$$

Expectation:

$$E[X] = \begin{cases} \int_{-\infty}^\infty x f_X(x) dx \\ \sum_i x_i p_X[x_i] \end{cases} \tag{17a}$$

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x) dx \\ \sum_i g(x_i) p_X[x_i] \end{cases} \tag{17b}$$

$$E[aX+b] = aE[x]+b \tag{17c}$$

Variance:

$$\begin{aligned} \operatorname{Var}[X] =& E\Big[(X-E[X])^2\Big] \\ =& E[X^2] - E^2[X] \end{aligned} \tag{18a}$$

$$\operatorname{Var}[aX+b] = a^2 \operatorname{Var}[X] \tag{18b}$$

$$\operatorname{Var}[b] = 0 \tag{18c}$$

3 Two Random Variables

3.1 Joint Distributions

Definitions:

$$F_{XY}(x,y) = p(X \le x, Y \le y) \tag{19a}$$

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \geqslant 0 \tag{19b}$$

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(s,p)\, dp\, ds \tag{19c}$$

$$p[x_j, y_k] = p(X = x_j, Y = y_k) \tag{20a}$$

$$F_{XY}(x,y) = p(X \leqslant x_j, Y \leqslant y_k) \tag{20b}$$

Expectation:

$$E[g(X,Y)] = \begin{cases} \iint g(x,y) f_{XY}(x,y) dx dy \\ \sum_i \sum_k g(x_i, y_k) p_X[x_i, y_k] \end{cases} \tag{21a}$$

$$E[aX+bY] = aE[X] + bE[Y] \tag{21b}$$

For **independent** random variables:

$$f_{XY}(x,y) = f_X(x) f_Y(y) \tag{22a}$$

$$p_{XY}[x_k, y_j] = p_X[x_k] p_Y[y_j] \tag{22b}$$

$$F_{XY}(x,y) = F_X(x) F_Y(y) \tag{22c}$$

$$E[XY] = E[X]E[Y] \tag{22d}$$

$$E[g_1(X) g_2(Y)] = E[g_1(X)] E[g_2(Y)] \tag{22e}$$

$$\operatorname{Var}[aX+bY] = a^2 \operatorname{Var}[X] + b^2 \operatorname{Var}[Y] \tag{22f}$$

Marginal distribution:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) dy \tag{23a}$$

$$p_X[x_k] = \sum_j p_{XY}[x_k, y_j] \tag{23b}$$

$$F_X(x) = F_{XY}(x, \infty) \tag{23c}$$

$$F_Y(y) = F_{XY}(\infty, y) \tag{23d}$$

3.2 Correlation, Covariance & Correlation Coefficient

- For two jointly-distributed random variables X and Y, covariance is given by

$$\begin{aligned} \operatorname{Cov}[X,Y] =& E\Big[(X-E[X])(Y-E[Y])\Big] \\ =& E[XY] - E[X]E[Y]. \end{aligned} \tag{24}$$

  Main covariance properties are:

$$\operatorname{Cov}[X,X] = Var[X] \tag{25a}$$

$$\operatorname{Cov}[X,Y] = \operatorname{Cov}[Y,X] \tag{25b}$$

$$\operatorname{Cov}[X,a] = 0 \tag{25c}$$

$$\operatorname{Cov}[aX,bY] = ab \operatorname{Cov}[X,Y] \tag{25d}$$

$$\operatorname{Cov}[X,Y] = \operatorname{Cov}[X+a, Y+b] \tag{25e}$$

$$\operatorname{Var}[X \pm Y] = \operatorname{Var}[X] + \operatorname{Var}[Y] \pm 2 \operatorname{Cov}[X,Y] \tag{25f}$$

$$\big|E[XY]\big| \le \sqrt{E\Big[X^2\Big] E\Big[Y^2\Big]} \text{ Cauchy-Schwatz} \tag{25g}$$

- Correlation coefficient (also termed as Pearson product-moment correlation coefficient) is given by

$$\rho_{XY} = \frac{\operatorname{Cov}[X,Y]}{\sqrt{\operatorname{Var}[X] \operatorname{Var}[Y]}} \tag{26}$$

  such that $|\rho_{XY}| \le 1$.

3.3 MMSE Linear Prediction

Mean square error (MSE) of predictor $\hat{Y}$ is given by

$$mse = E[(Y - \hat{Y})^2] \tag{27}$$

Linear prediction of $\hat{Y} = ax + b$ for $X = x$ is

$$\hat{Y} = E[Y] + \frac{\operatorname{Cov}[X,Y]}{\operatorname{Var}[X]} (x - E[X]) \tag{28}$$

and

$$mse_{min} = E\left[\Big(Y-(aX+b)\Big)^2\right] = \mathrm{Var}(Y)(1-\rho_{XY}^2) \tag{29}$$

When X, Y are jointly Gaussian, this prediction is optimal among **all** possible predictors

3.4 Relations

- When X and Y are *orthogonal*, $E[XY] = 0$.
- When X and Y are *uncorrelated*, $\mathrm{Cov}[X,Y] = \rho_{XY} = 0$.
- When X and Y are *independent*, they are also uncorrelated (see also Eqs. 22).
- When X and Y are *jointly* Gaussian and uncorrelated $\Rightarrow$ X and Y are independent.

3.5 Bi-variate Normal Distribution

Joint Gaussian distribution of X_1 and X_2

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \mathrm{C}_{\mathbf{X}}\right) \tag{30}$$

with covariance matrix

$$\mathrm{C}_{\mathbf{X}} = \begin{bmatrix} \mathrm{Cov}[X_1,X_1] & \mathrm{Cov}[X_1,X_2] \\ \mathrm{Cov}[X_2,X_1] & \mathrm{Cov}[X_2,X_2] \end{bmatrix} \tag{31}$$

Important properties:

- Sum of independent Gaussian variables is a Gaussian variable.
- Random vector $[X_1,\dots,X_n]$ is **jointly** Gaussian distributed, iff (if and only if) for all possible real vectors $\mathbf{a} = (a_1,\dots,a_n)^T$ linear combination $Y = a_1X_1 + \cdot + a_nX_n$ is Gaussian distributed.
- If jointly distributed Gaussian random variables are *uncorrelated*, they are also *independent*

4 Random Processes – General Properties

- PDF & CDF

$$F_{\mathbf{x}}(x;t) = p(\mathbf{x}(t) \le x) \tag{32a}$$
$$f_{\mathbf{x}}(x;t) = \frac{\partial}{\partial x} F_{\mathbf{x}}(x;t) \tag{32b}$$
$$p_{\mathbf{x}}[x_k;n] = p(\mathbf{x}[n] = x_k) \tag{32c}$$

- Average:

$$E[\mathbf{x}(t)] = \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x;t)dx \tag{33a}$$
$$E\Big[\mathbf{x}[n]\Big] = \sum_i x_i p_{\mathbf{x}}[x_k;n] \tag{33b}$$

- Variance:

$$\mathrm{Var}\left[\mathbf{x}(t)\right] = E\left[\mathbf{x}^2(t)\right] - E^2\left[\mathbf{x}(t)\right] = \sigma_{\mathbf{x}}(t) \tag{34a}$$
$$\mathrm{Var}\left[\mathbf{x}[n]\right] = E\left[\mathbf{x}^2[n]\right] - E^2\left[\mathbf{x}[n]\right] = \sigma_{\mathbf{x}}[n] \tag{34b}$$

- Auto-correlation

$$R_{\mathbf{x}}(t_1,t_2) = E[\mathbf{x}(t_1)\mathbf{x}(t_2)] \tag{35a}$$
$$R_{\mathbf{x}}(t,t+\tau) = E[\mathbf{x}(t)\mathbf{x}(t+\tau)] \tag{35b}$$
$$R_{\mathbf{x}}(t_1,t_2) = R_{\mathbf{x}}(t_2,t_1) \tag{35c}$$
$$R_{\mathbf{x}}(t,t) = E[\mathbf{x}^2(t)] \tag{35d}$$
$$R_{\mathbf{x}}[n_1,n_2] = E\left[\mathbf{x}[n_1]\mathbf{x}[n_2]\right] \tag{35e}$$
$$R_{\mathbf{x}}[n,n] = E\left[\mathbf{x}^2[n]\right] \tag{35f}$$

- Auto-covariance

$$C_{\mathbf{x}}(t_1,t_2) = E\Big[\big\{\mathbf{x}(t_1) - E[\mathbf{x}(t_1)]\big\}\big\{\mathbf{x}(t_2) - E[\mathbf{x}(t_2)]\big\}\Big] \tag{36}$$
$$= R_{\mathbf{x}}(t_1,t_2) - E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)] \tag{37}$$
$$C_{\mathbf{x}}[n_1,n_2] = E\Big[\big\{\mathbf{x}[n_1] - E[\mathbf{x}[n_1]]\big\}\big\{\mathbf{x}[n_2] - E[\mathbf{x}[n_2]]\big\}\Big] \tag{38}$$
$$= R_{\mathbf{x}}[n_1,n_2] - E\left[\mathbf{x}[n_1]\right]E\left[\mathbf{x}[n_2]\right] \tag{39}$$

$$C_{\mathbf{x}}(t,t) = \mathrm{Var}[\mathbf{x}(t)] \tag{40a}$$
$$C_{\mathbf{x}}[n,n] = \mathrm{Var}[\mathbf{x}[n]] \tag{40b}$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(t_1,t_2) = \frac{C_{\mathbf{x}}(t_1,t_2)}{\sqrt{C_{\mathbf{x}}(t_1,t_1)C_{\mathbf{x}}(t_2,t_2)}} \tag{41a}$$

$$\left|\rho_{\mathbf{x}}(t_1,t_2)\right| \leqslant 1 \tag{41b}$$

- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *orthogonal*, $R_{\mathbf{x}}(t_1,t_2)=0$.
- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *uncorrelated*, $C_{\mathbf{x}}(t_1,t_2)=\rho_{\mathbf{x}}(t_1,t_2)=0$.
- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *independent*, $R_{\mathbf{x}}(t_1,t_2)=E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)]$.

5 Wide-Sense Stationary (WSS) Process

Definition:

$$E[\mathbf{x}(t)] = E[\mathbf{x}(0)] = \mu_{\mathbf{x}} = \text{const} \tag{42a}$$
$$R_{\mathbf{x}}(t_1,t_2) = R_{\mathbf{x}}(\tau = |t_2-t_1|), \quad \forall t_1,t_2 \tag{42b}$$
$$E[\mathbf{x}[n]] = E[\mathbf{x}[0]] = \mu_{\mathbf{x}} = \text{const} \tag{42c}$$
$$R_{\mathbf{x}}[n_1,n_2] = R_{\mathbf{x}}(k = |n_2-n_1|), \quad \forall n_1,n_2 \tag{42d}$$

- Auto-correlation

$$R_{\mathbf{x}}(\tau) = E[\mathbf{x}(t)\mathbf{x}(t+\tau)] \tag{43a}$$
$$R_{\mathbf{x}}[k] = E\left[\mathbf{x}[n]\mathbf{x}[n+k]\right] \tag{43b}$$

Properties:

$$R_{\mathbf{x}}(-\tau) = R_{\mathbf{x}}(\tau) \tag{44a}$$
$$R_{\mathbf{x}}(0) = E[\left|\mathbf{x}(0)\right|^2] = E[\left|\mathbf{x}(t)\right|^2] \tag{44b}$$
$$Var[\mathbf{x}(t)] = C_{\mathbf{x}}(0) = \sigma_{\mathbf{x}}^2 \tag{44c}$$
$$R_{\mathbf{x}}(0) \geqslant \left|R_{\mathbf{x}}(\tau)\right| \tag{44d}$$

- Auto-covariance

$$C_{\mathbf{x}}(\tau) = R_{\mathbf{x}}(\tau) - \mu_{\mathbf{x}}^2 \tag{45a}$$
$$C_{\mathbf{x}}[k] = R_{\mathbf{x}}[k] - \mu_{\mathbf{x}}^2 \tag{45b}$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(\tau) = \frac{C_{\mathbf{x}}(\tau)}{C_{\mathbf{x}}(0)} \tag{46a}$$
$$\rho_{\mathbf{x}}[k] = \frac{C_{\mathbf{x}}[k]}{C_{\mathbf{x}}[0]} \tag{46b}$$

5.1 Power Spectral Density (PSD)

$$S_{\mathbf{x}}(F) = \mathcal{F}\left\{R_{\mathbf{x}}(\tau)\right\} = \qquad -\infty \leq F \leq \infty$$
$$= \int_{-\infty}^{\infty} R_{\mathbf{x}}(\tau)\exp\left(-j2\pi F\tau\right)d\tau \tag{47a}$$
$$R_{\mathbf{x}}(\tau) = \mathcal{F}^{-1}\left\{S_{\mathbf{x}}(F)\right\} =$$
$$= \int_{-\infty}^{\infty} S_{\mathbf{x}}(f)\exp\left(j2\pi F\tau\right)dF \tag{47b}$$
$$S_{\mathbf{x}}(f) = \text{DTFT}\left\{R_{\mathbf{x}}[k]\right\} = \sum_{k=-\infty}^{\infty} R_{\mathbf{x}}[k]e^{-j2\pi fk} \tag{47c}$$

Properties:

$$S_{\mathbf{x}}(F) = S_{\mathbf{x}}(-F) \tag{48a}$$
$$S_{\mathbf{x}}(F) \geqslant 0,\ \forall F \tag{48b}$$
$$S_{\mathbf{x}}(F) \in \mathbb{R} \tag{48c}$$
$$S_{\mathbf{x}}(f) = S_{\mathbf{x}}(-f) \tag{48d}$$
$$S_{\mathbf{x}}(f) \geqslant 0,\ \forall f \tag{48e}$$
$$S_{\mathbf{x}}(f) \in \mathbb{R} \tag{48f}$$
$$S_{\mathbf{x}}(f) = S_{\mathbf{x}}(f+1) \tag{48g}$$

Average power

$$P_{\mathbf{x}} = E\left[\mathbf{x}^2(t)\right] = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(F)dF \tag{49a}$$
$$P_{\mathbf{x}} = E\left[\mathbf{x}^2[n]\right] = R_{\mathbf{x}}[0] = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{\mathbf{x}}(f)df \tag{49b}$$

5.2 White Noise & White Gaussian Noise (WGN) Process

White noise process is SSS (WSS) process that is characterized by

$$R_{\mathbf{n}}(\tau) = \sigma^2\delta(\tau) \tag{50a}$$
$$S_{\mathbf{n}}(F) = \sigma^2 \quad \forall F \tag{50b}$$

For WGN process, $\mathbf{n}(t) \sim N(0,\sigma^2)$,

$$R_{\mathbf{n}}(\tau) = \frac{N_0}{2}\delta(\tau) \tag{51a}$$
$$S_{\mathbf{n}}(F) = \frac{N_0}{2} \quad \forall F \tag{51b}$$

5.3 Relation Between Covariance Matrix & Auto-covariance

Given WSS process $\mathbf{x}(t)$, the corresponding correlation matrix of $\mathbf{X} = [\mathbf{x}(t_1),\ldots,\mathbf{x}(t_N)]^T$ is given by

$$R_{\mathbf{X}} = E\left[\mathbf{X}\mathbf{X}^T\right] \tag{52}$$
$$R_{\mathbf{X}}(i,j) = E\left[X_iX_j\right] = R_{\mathbf{x}}\left(|t_i-t_j|\right) \tag{53}$$

6 Cross-Signal

- Cross-correlation

$$R_{\mathbf{xy}}(t_1,t_2)=E[\mathbf{x}(t_1)\mathbf{y}(t_2)] \tag{54}$$

- Cross-covariance

$$C_{\mathbf{xy}}(t_1,t_2)=R_{\mathbf{xy}}(t_1,t_2)-E[\mathbf{x}(t_1)]E[\mathbf{y}(t_2)] \tag{55}$$

- Correlation Coefficient

$$\rho_{\mathbf{xy}}(t_1,t_2)=\frac{C_{\mathbf{xy}}(t_1,t_2)}{\sqrt{C_{\mathbf{x}}(t_1,t_1)C_{\mathbf{y}}(t_2,t_2)}} \tag{56}$$

6.1 WSS Cross-signal

- $\mathbf{x}(t),\mathbf{y}(t)$ are jointly WSS, if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ each of them is WSS and

$$R_{\mathbf{xy}}(\tau)=E[\mathbf{x}(t)\mathbf{y}(t+\tau)] \tag{57}$$

- When $\mathbf{x}(t)$ and $\mathbf{y}(t+\tau)$ are *uncorrelated jointly WSS*, $C_{\mathbf{xy}}(\tau)=0$.

  Properties

$$R_{\mathbf{xy}}(\tau)=R_{\mathbf{yx}}(-\tau) \tag{58a}$$

$$\left|R_{\mathbf{xy}}(\tau)\right|\leqslant\sqrt{R_{\mathbf{x}}(0)R_{\mathbf{y}}(0)} \tag{58b}$$

$$\left|R_{\mathbf{xy}}(\tau)\right|\leqslant\frac{1}{2}\left[R_{\mathbf{x}}(0)+R_{\mathbf{y}}(0)\right] \tag{58c}$$

- Cross-covariance

$$C_{\mathbf{xy}}(\tau)=R_{\mathbf{xy}}(\tau)-\mu_{\mathbf{x}}\mu_{\mathbf{y}} \tag{59}$$

- Cross-PSD

$$S_{\mathbf{xy}}(f)=\mathcal{F}\left\{R_{\mathbf{xy}}(\tau)\right\} \tag{60}$$

  Properties

$$S_{\mathbf{xy}}(f)=S_{\mathbf{yx}}(-f)=S^*_{\mathbf{xy}}(-f) \tag{61}$$

  Correlation coefficient

$$\rho_{\mathbf{xy}}(\tau)=\frac{C_{\mathbf{xy}}(\tau)}{C_{\mathbf{xy}}(0)} \tag{62}$$

- Coherence

$$\gamma_{\mathbf{xy}}(f)=\frac{S_{\mathbf{xy}}(f)}{\sqrt{S_{\mathbf{x}}(f)S_{\mathbf{y}}(f)}} \tag{63}$$

7 LTI and WSS Random Process

Output of LTI system with impulse response $h(t)$ and random process $x(t)$,

$$y(t)=x(t)*h(t) \tag{64}$$

Average

$$m_{\mathbf{y}}=m_{\mathbf{x}}\int_{-\infty}^{\infty}h(s)ds=m_{\mathbf{x}}H(f=0) \tag{65}$$

Cross-correlation & cross-covariance:

$$R_{\mathbf{xy}}(\tau)=R_{\mathbf{x}}(\tau)*h(\tau) \tag{66a}$$
$$C_{\mathbf{xy}}(\tau)=C_{\mathbf{x}}(\tau)*h(\tau) \tag{66b}$$
$$R_{\mathbf{yx}}(\tau)=R_{\mathbf{x}}(\tau)*h(-\tau) \tag{66c}$$
$$C_{\mathbf{yx}}(\tau)=C_{\mathbf{x}}(\tau)*h(-\tau) \tag{66d}$$
$$R_{\mathbf{y}}(\tau)=R_{\mathbf{x}}(\tau)*h(\tau)*h(-\tau) \tag{66e}$$
$$C_{\mathbf{y}}(\tau)=C_{\mathbf{x}}(\tau)*h(\tau)*h(-\tau) \tag{66f}$$

Power-Spectral Density (PSD) & Cross-PSD: Given frequency response
$H(F)=\mathcal{F}\left\{h(\tau)\right\},\ H^*(F)=\mathcal{F}\left\{h(-\tau)\right\}$

$$S_{\mathbf{xy}}(F)=S_{\mathbf{x}}(F)H(F) \tag{67a}$$
$$S_{\mathbf{yx}}(F)=S_{\mathbf{x}}(F)H^*(F) \tag{67b}$$
$$S_{\mathbf{y}}(F)=S_{\mathbf{x}}(F)H(F)H^*(F)=S_{\mathbf{x}}(F)\left|H(F)\right|^2 \tag{67c}$$

Power of the process:

$$P_x=R_{\mathbf{x}}(0)=\int_{-\infty}^{\infty}S_{\mathbf{x}}(F)\,dF \tag{68a}$$
$$P_y=R_{\mathbf{y}}(0)=\int_{-\infty}^{\infty}S_{\mathbf{x}}(F)\left|H(F)\right|^2dF \tag{68b}$$
$$P_x=R_{\mathbf{x}}[0]=\int_{-1/2}^{1/2}S_{\mathbf{x}}(f)\,df \tag{68c}$$
$$P_y=R_{\mathbf{y}}[0]=\int_{-1/2}^{1/2}S_{\mathbf{x}}(f)\left|H(f)\right|^2df \tag{68d}$$

Same process passes two different systems

$$R_{\mathbf{yz}}(\tau)=R_{\mathbf{x}}(\tau)*h_1(-\tau)*h_2(\tau) \tag{69}$$
$$S_{\mathbf{yz}}(F)=S_{\mathbf{x}}(F)H_1^*(F)H_2(F) \tag{70}$$

7.1 Z-Transform

Auto-correlation

$$
\begin{aligned}
H(z) =& \mathscr{Z}\left\{h[n]\right\} = \frac{B(z)}{A(z)} \\
&\mathscr{Z}\left\{h[n] * h[-n]\right\} = \frac{B(z)B(z^{-1})}{A(z)A(z^{-1})} \\
S_{\mathbf{x}}(z) =& \mathscr{Z}\left\{R_{\mathbf{x}}[n]\right\}
\end{aligned}
$$

PSD

$$S_{\mathbf{xy}}(z) = S_{\mathbf{x}}(z)H(z) \tag{71a}$$

$$S_{\mathbf{yx}}(z) = S_{\mathbf{x}}(z)H(z^{-1}) \tag{71b}$$

$$S_{\mathbf{y}}(z) = S_{\mathbf{x}}(z)H(z)H(z^{-1}) \tag{71c}$$

Two different systems

$$R_{\mathbf{yz}}[k] = R_{\mathbf{x}}[k] * h_1[-k] * h_2[k] \tag{72a}$$

$$S_{\mathbf{yz}}(f) = S_{\mathbf{x}}(f)H_1^*(f)H_2(f) \tag{72b}$$

$$S_{\mathbf{yz}}(z) = S_{\mathbf{x}}(z)H_1(1/z)H_2(z) \tag{72c}$$

7.2 Gaussian Process

A Gaussian process $\mathbf{x}(t)$ a random process that for $\forall k > 0$ and for all times $t_1, \ldots, t_k$, the set of random variable $\mathbf{x}(t_1), \ldots, \mathbf{x}(t_k)$ is jointly Gaussian.

Properties:

- WSS Gaussian process is SSS.
- Gaussian process $\mathbf{x}(t)$ that passes through LTI system, $\mathbf{y}(t) = h(t) * \mathbf{x}(t)$, is also Gaussian process that may be described by the change of expectation and auto-correlation,

$$E[\mathbf{y}(t)] = E[\mathbf{x}(t)] \int_{-\infty}^{\infty} h(s)ds \tag{73a}$$

$$= E[\mathbf{x}(t)]H(0), \quad H(F) = \mathscr{F}\left\{h(t)\right\}$$

$$C_{\mathbf{y}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \tag{73b}$$

- The resulting autocorrelation may be used for producing the correspondent covariance matrix $C_{\mathbf{Y}}$ of a multivariate Gaussian $\mathbf{Y} = [\mathbf{y}(t_1), \ldots, \mathbf{y}(t_N)]^T$

7.3 Linear Prediction

Given N samples of process $\mathbf{x}[n]$, and predictor

$$\hat{\mathbf{x}}[n+1] = \sum_{i=1}^{N} a_i \mathbf{x}[n-i+1], \tag{74}$$

the mean-square error is given by

$$
\begin{aligned}
mse &= E\left[\left(\mathbf{x}[n+1] - \hat{\mathbf{x}}[n+1]\right)^2\right] \\
&= E\left[\left(\mathbf{x}[n+1] - a_0\mathbf{x}[n] - a_1\mathbf{x}[n-1] - \ldots - a_N\mathbf{x}[n-N]\right)^2\right]
\end{aligned} \tag{75}
$$

and the values of a_i are given by a solution of

$$
\begin{bmatrix}
R_{\mathbf{x}}[0] & R_{\mathbf{x}}[1] & \cdots & R_{\mathbf{x}}[N-1] \\
R_{\mathbf{x}}[1] & R_{\mathbf{x}}[0] & \cdots & R_{\mathbf{x}}[N-2] \\
\vdots & \vdots & \ddots & \vdots \\
R_{\mathbf{x}}[N-1] & R_{\mathbf{x}}[N-2] & \cdots & R_{\mathbf{x}}[0]
\end{bmatrix}
\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}
=
\begin{bmatrix} R_{\mathbf{x}}[1] \\ R_{\mathbf{x}}[2] \\ \vdots \\ R_{\mathbf{x}}[N] \end{bmatrix} \tag{76}
$$

and the resulting minimum MSE is

$$mse_{min} = R_{\mathbf{x}}[0] - \sum_{i=1}^{N} a_i R_{\mathbf{x}}[i] \tag{77}$$

8 Different Supplementary Formulas

8.1 Derivatives

$$\frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d}{dx}\exp\left[f(x)\right] = \exp\left[f(x)\right]\frac{d}{dx}f(x)$$

8.2 Integrals

8.2.1 Indefinite

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1$$
$$\int \exp(ax)dx = \frac{1}{a}\exp(ax)$$
$$\int x\exp(ax)dx = \exp(ax)\left[\frac{x}{a} - \frac{1}{a^2}\right]$$
$$\int x^2\exp(ax)dx = \exp(ax)\left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right]$$
$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

8.2.2 Definite

$$\int_0^\infty \exp\Big(-a^2x^2\Big)dx = \frac{\sqrt{\pi}}{2a}$$
$$\int_0^\infty x^2\exp\Big(-a^2x^2\Big)dx = \frac{\sqrt{\pi}}{4a^3}$$
$$\int_{-\infty}^{\infty}\delta(x)dx = 1$$
$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

8.3 Fourier Transform

8.3.1 Properties

$$\frac{d^n}{dt^n}f(t) \overset{\mathscr{F}}{\Longleftrightarrow} (j2\pi f)^n F(f)$$
$$f(-t) \overset{\mathscr{F}}{\Longleftrightarrow} F^*(f)$$
$$f(t-t_0) \overset{\mathscr{F}}{\Longleftrightarrow} F(f)e^{-j2\pi f t_0}$$
$$f(t)e^{j2\pi f_0 t} \overset{\mathscr{F}}{\Longleftrightarrow} F(f-f_0)$$

8.3.2 Transform pairs

$$u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2}\left(\frac{1}{j\pi f} + \delta(f)\right)$$
$$\exp(-at)u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{a+j2\pi f}$$
$$t\exp(-at)u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{(a+j2\pi f)^2}$$
$$\exp\big(-a|t|\big) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{2a}{a^2+4\pi^2 f^2}$$
$$\exp\Big(-at^2\Big) \overset{\mathscr{F}}{\Longleftrightarrow} \sqrt{\frac{\pi}{a}}\exp\left(-\frac{(\pi f)^2}{a}\right)$$
$$\cos(2\pi f_a t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2}\Big[\delta(f-f_a) + \delta(f+f_a)\Big]$$
$$\sin(2\pi f_a t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2j}\Big[\delta(f-f_a) - \delta(f+f_a)\Big]$$

8.4 Convolution

$$x(t)*y(t) = \int_{-\infty}^{\infty} f(s)g(t-s)ds$$
$$x(t)*y(t) \overset{\mathscr{F}}{\Longleftrightarrow} X(f)Y(f)$$
$$\delta(t)*y(t) = y(t)$$

8.5 Trigonometry

$$\sin^2(\alpha) = \frac{1}{2}\big(1-\cos(2\alpha)\big)$$
$$\cos^2(\alpha) = \frac{1}{2}\big(1+\cos(2\alpha)\big)$$
$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$
$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\Big[\sin(\alpha-\beta) + \sin\big((\alpha+\beta)\big)\Big]$$

8.6 Matrices

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
$$\det[\mathbf{A}] = ad - bc$$
$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

9 Z-transforms

$$X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$

9.1 Usual Transforms

Signal	Z transform	ROC
$\delta[n]$	1	$\mathbb{C}$
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	$\mathbb{C} - \{0\}$ if $m > 0$, $\mathbb{C} - \{\infty\}$ if $m < 0$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < a$

9.2 Properties

Property	Discrete Signal	Z transform	ROC
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(z) + a_2 X_2(z)$	includes $R_1 \cap R_2$
Time shift	$x[n-n_0]$	$z^{-n_0} X(z)$	R
Frequency scaling	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$\lvert z_0\rvert R$
Time reversal	$x[-n]$	$X(z^{-1})$	R^{-1} if $m < 0$
Convolution	$(x_1 * x_2)[n]$	$X_1(z) X_2(z)$	$R_1 \cap R_2$ (or possibly more)
Time differentiation	$x[n] - x[n-1]$	$(1-z^{-1}) X(z)$	$R \cap \{\lvert z\rvert > 0\}$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{X(z)}{1-z^{-1}}$	$R \cap \{\lvert z\rvert > 1\}$