המחלקה להנדסת חשמל ואלקטרוניקה

תאריך הבחינה: 23.02.20

שעות הבחינה: 13:30-16:30

מבוא לאותות אקראיים

מועד ב'

ד"ר דימה בחובסקי

תשע"ט סמסטר א'

חומר עזר - דף נוסחאות אישי (משני צדדים), מחשבון

הוראות מיוחדות:

- סעיפים הם בעלי ניקוד זהה, אלא אם צוין אחרת.
- יש לציין באופן מלא וברור את שלבי הפתרון. תשובה ללא הסבר לא תתקבלנה.
- במקום בו נדרש חישוב מספרי, יש קודם לרשום את הנוסחא, ורק אח"כ להציב!
- יש לציין יחידות למספרים, ובמידה וקיימות!
- כל השרטוטים יהיו גדולים, ברורים, עם סימון צירים!
- אין חובה להגיע לערך מספרי של הפונקציה $Q(x)$, במידה ומופיעה בתשובה.

השאלון כולל 10 דפים (כולל דף זה)

בהצלחה!

1 קשר בין משתנים (15 נק')

נתונים קשרים הבאים בין המשתנים X, Y:

$$\mathrm{Var}\left[X+2Y\right]=40 \tag{1}$$

$$\mathrm{Var}\left[X-2Y\right]=20 \tag{2}$$

$$\mathrm{Var}\left[X\right]=\mathrm{Var}\left[Y\right] \tag{3}$$

חשב מספרית את כל האיברים של מטריצת covariance בין המשתנים X, Y.

2 תהליכים בזמן רציף (90 נק')

נתונים תהליכים $\mathbf{x}(t), \mathbf{y}(t)$.

$\mathbf{x}(t)$ בעל מאפיינים הבאים:

- גאוסי
- סטציאונרי
- $E\left[\mathbf{x}(t)\right]=0$
- $R_{\mathbf{x}}(\tau)=\exp\left(-\frac{|\tau|}{4}\right)$

$$\mathbf{y}(t)=\mathbf{x}(t)+\cos(2\pi 30t)$$

1. חשב הספק הממוצע של $\mathbf{x}(t)$.
2. חשב הספק ממוצע של $\mathbf{x}(t)$ עבור תחום תדרים של $|F|<4\mathrm{Hz}$.
3. חשב הסתברות $\Pr\Big(\mathbf{x}(1)>3\Big)$.
4. חשב הסתברות $\Pr\Big(\left[\mathbf{x}(0)+\mathbf{x}(1)\right]>3\Big)$.
5. חשב $\mathrm{Var}\left[\mathbf{y}(t)\right]$ והספק ממוצע של $\mathbf{y}(t)$.
6. חשב $R_{\mathbf{y}}(\tau)$, $E\left[\mathbf{y}(t)\right]$. האם מדובר בתהליך WSS?

Random Processes – Formulas

1 Distributions

1.1 Continuous

	Notation	PDF	CDF	$E[X]$	$\mathrm{Var}[X]$
Uniform	U[a,b]	$\begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & b < x \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal	$N(\mu,\sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$	$\Phi(x)$	μ	σ^2
Exponential	$Exp(\lambda)$	$\lambda \exp(-\lambda x), x \ge 0$	$1-\exp(-\lambda x)$	$1/\lambda$	$1/\lambda^2$

1.1.1 Q-function

Given $Y \sim N(\mu,\sigma^2)$

$$p(Y > y) = Q\left(\frac{y-\mu}{\sigma}\right) \tag{1a}$$

$$Q(x) = 1 - \Phi(x) \tag{1b}$$

$$Q(-x) = 1 - Q(x) \tag{1c}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{s^2}{2}\right) ds. \tag{2}$$

1.2 Discrete

	Notation	PMF	CDF	$E[X]$	$\mathrm{Var}[X]$
Bernoulli	Ber(p)	$\begin{cases} 1-p & k=0 \\ p & k=1 \end{cases}$	$\begin{cases} 0 & x < 0 \\ 1-p & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$	p	$p(1-p)$

2 Random Variables

Definitions:

$$F_X(x) = p(X \le x) \tag{3a}$$

$$f_X(x) = \frac{\partial F_X(x)}{\partial x} \ge 0 \tag{3b}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(p)\, dp \tag{3c}$$

$$p(a < X \leqslant b) = F_X(b) - F_X(a) \tag{3d}$$

$$f_X(x) \ge 0 \tag{3e}$$

$$\int_{-\infty}^{\infty} f_X(x) dx = 1 \tag{3f}$$

$$p_X[x_k] = \Pr[X = x_k] \tag{4a}$$

$$0 \le p_X[x_i] \le 1 \ \ \forall i \tag{4b}$$

$$\sum_i p_X[x_i] = 1 \tag{4c}$$

$$F_X(x) = \Pr(X \le x),\ \ x \in \mathbb{R} \tag{4d}$$

$$F_X(x) = \sum_{k: x_k \leqslant x} p_X[x_k] \tag{4e}$$

Expectation:

$$E[X] = \begin{cases} \int_{-\infty}^{\infty} x f_X(x) dx \\ \sum_i x_i p_X[x_i] \end{cases} \tag{5a}$$

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x) dx \\ \sum_i g(x_i) p_X[x_i] \end{cases} \tag{5b}$$

$$E[aX+b] = aE[x]+b \tag{5c}$$

Variance:

$$\begin{aligned} \mathrm{Var}[X] =& E\big[(X-E[X])^2\big] \\ =& E[X^2]-E^2[X] \end{aligned} \tag{6a}$$

$$\mathrm{Var}[aX+b] = a^2\,\mathrm{Var}[X] \tag{6b}$$

$$\mathrm{Var}[b] = 0 \tag{6c}$$

3 Two Random Variables

3.1 Joint Distributions

Definitions:

$$F_{XY}(x,y) = p(X \le x, Y \le y) \tag{7a}$$

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \geqslant 0 \tag{7b}$$

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(s,p)\,dp\,ds \tag{7c}$$

$$p[x_j, y_k] = p(X = x_j, Y = y_k) \tag{8a}$$

$$F_{XY}(x,y) = p(X \leqslant x_j, Y \leqslant y_k) \tag{8b}$$

Expectation:

$$E[g(X,Y)] = \begin{cases} \iint g(x,y) f_{XY}(x,y) dx dy \\ \sum_i \sum_k g(x_i,y_k) p_X[x_i,y_k] \end{cases} \tag{9a}$$

$$E[aX+bY] = aE[X]+bE[Y] \tag{9b}$$

For **independent** random variables:

$$f_{XY}(x,y) = f_X(x) f_Y(y) \tag{10a}$$

$$p_{XY}[x_k,y_j] = p_X[x_k] p_Y[y_j] \tag{10b}$$

$$F_{XY}(x,y) = F_X(x) F_Y(y) \tag{10c}$$

$$E[XY] = E[X]E[Y] \tag{10d}$$

$$E[g_1(X)g_2(Y)] = E[g_1(X)]E[g_2(Y)] \tag{10e}$$

$$\mathrm{Var}[aX+bY] = a^2\,\mathrm{Var}[X] + b^2\,\mathrm{Var}[Y] \tag{10f}$$

Marginal distribution:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) dy \tag{11a}$$

$$p_X[x_k] = \sum_j p_{XY}[x_k,y_j] \tag{11b}$$

$$F_X(x) = F_{XY}(x,\infty) \tag{11c}$$

$$F_Y(y) = F_{XY}(\infty,y) \tag{11d}$$

3.2 Conditional Relations

Conditional distribution (Bayes),
for $f_X(x), f_Y(y), p_X[x_k], p_Y[y_k] > 0$:

$$f_{Y|X}(y|x) f_X(x) = f_{X|Y}(x|y) f_Y(x) = f_{XY}(x,y) \tag{12a}$$

$$p_{Y|X}[y_j|x_k] p_X[x_k] = p_{X|Y}[x_k|y_j] p_Y[y_j] = p_{XY}[x_k,y_j] \tag{12b}$$

$$F_{Y|X}(y|x) = p(Y \leqslant y | X = x) \tag{12c}$$

$$= \int_{-\infty}^{y} f_{Y|X}(s|x) ds \tag{12d}$$

$$F_{Y|X}[y|x_k] = \frac{p[Y \leqslant y_j, X = x_k]}{p_X[x_k]} \tag{12e}$$

Conditional expectation & Variance:

$$E[Y|X] = \begin{cases} \int y f_{Y|X}(y|x) dy \\ \sum_j y_j p[y_j|x_k] \end{cases} \tag{13a}$$

$$E[X] = E\big[E[X|Y]\big] = \iint y f_{Y|X}(y|x) f_X(x) dx dy \tag{13b}$$

$$\mathrm{Var}[Y|X] = E[Y^2|X] - E^2[Y|X] \tag{13c}$$

$$\mathrm{Var}[Y] = \mathrm{Var}\big[E[Y|X]\big] + E\big[\mathrm{Var}[Y|X]\big] \tag{13d}$$

3.3 Correlation, Covariance & Correlation Coefficient

- For two jointly-distributed random variables X and Y, covariance is given by

$$\begin{aligned} \mathrm{Cov}[X,Y] =& E\big[(X-E[X])(Y-E[Y])\big] \\ =& E[XY] - E[X]E[Y]. \end{aligned} \tag{14}$$

Main covariance properties are:

$$\mathrm{Cov}[X,X] = Var[X] \tag{15a}$$

$$\mathrm{Cov}[X,Y] = \mathrm{Cov}[Y,X] \tag{15b}$$

$$\mathrm{Cov}[X,a] = 0 \tag{15c}$$

$$\mathrm{Cov}[aX,bY] = ab\,\mathrm{Cov}[X,Y] \tag{15d}$$

$$\mathrm{Cov}[X,Y] = \mathrm{Cov}[X+a, Y+b] \tag{15e}$$

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X,Y] \tag{15f}$$

$$\big|E[XY]\big| \le \sqrt{E\big[X^2\big]E\big[Y^2\big]} \text{ Cauchy-Schwatz} \tag{15g}$$

- Correlation coefficient (also termed as Pearson product-moment correlation coefficient) is given by

$$\rho_{XY} = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} \tag{16}$$

such that $|\rho_{XY}| \le 1$.

3.4 MMSE Linear Prediction

Mean square error (MSE) of predictor $\hat{Y}$ is given by

$$mse = E[(Y-\hat{Y})^2] \tag{17}$$

Linear prediction of $\hat{Y} = ax+b$ for $X = x$ is

$$\hat{Y} = E[Y] + \frac{\mathrm{Cov}[X,Y]}{\mathrm{Var}[X]}\big(x - E[X]\big) \tag{18}$$

and

$$mse_{min} = E\left[\big(Y-(aX+b)\big)^2\right] = \mathrm{Var}(Y)(1-\rho_{XY}^2) \tag{19}$$

When X, Y are jointly Gaussian, this prediction is optimal among **all** possible predictors

3.5 Relations

- When X and Y are *orthogonal*, $E[XY] = 0$.
- When X and Y are *uncorrelated*, $\mathrm{Cov}[X,Y] = \rho_{XY} = 0$.
- When X and Y are *independent*, they are also uncorrelated (see also Eqs. 10).
- When X and Y are *jointly* Gaussian and uncorrelated $\Rightarrow$ X and Y are independent.

4 Multi-dimensional Random Variables

4.1 Covariance matrix

Given random vector $\mathbf{X} = (X_1, X_2, \ldots, X_N)^T$,

$$\begin{aligned} \mathrm{C}_{\mathbf{X}} &= \mathrm{Cov}[\mathbf{X},\mathbf{X}] = E[(\mathbf{X}-E[\mathbf{X}])(\mathbf{X}-E[\mathbf{X}])^T] \\ &= E[\mathbf{X}\mathbf{X}^T] - E[\mathbf{X}]E[\mathbf{X}]^T \\ &= \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1,X_2] & \cdots & \mathrm{Cov}[X_1,X_N] \\ \mathrm{Cov}[X_2,X_1] & \mathrm{Var}[X_2] & \cdots & \mathrm{Cov}[X_2,X_N] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}[X_N,X_1] & \mathrm{Cov}[X_N,X_2] & \cdots & \mathrm{Var}[X_N] \end{bmatrix} \end{aligned} \tag{20}$$

Properties:

- Symmetry

$$\mathrm{C}_{\mathbf{X}} = \mathrm{C}_{\mathbf{X}}^T \quad \mathrm{Cov}[X_i,X_j] = \mathrm{Cov}[X_j,X_i] \tag{21}$$

- Variance of linear combination: Given vector $\mathbf{a} = (a_1, a_2, \ldots, a_N)^T$,

$$\mathrm{Var}[\mathbf{a}^T\mathbf{X}] = \mathbf{a}^T C_{\mathbf{X}}\mathbf{a} \tag{22}$$

- Linear transformation: Given linear tranformation $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$,

$$E[\mathbf{Y}] = \mathbf{A}E[\mathbf{X}] + \mathbf{b} \tag{23a}$$

$$C_{\mathbf{Y}} = \mathbf{A}C_{\mathbf{X}}\mathbf{A}^T \tag{23b}$$

- Uncorrelated variables

$$\mathrm{C}_{\mathbf{X}} = \mathrm{diag}\big[\mathrm{Var}[X_1], \mathrm{Var}[X_2], \ldots, \mathrm{Var}[X_N]\big] \tag{24}$$

- Cross-covariance: For two random vectors $\mathbf{X} \in \mathbb{R}^m$ and $\mathbf{Y} \in \mathbb{R}^n$, the resulting $m \times n$ cross-covariance matrix is given by

$$\begin{aligned} \mathrm{Cov}[\mathbf{X},\mathbf{Y}] &= C_{\mathbf{XY}} \\ &= E[(\mathbf{X}-E[\mathbf{X}])(\mathbf{Y}-E[\mathbf{Y}])^T] \\ &= E[\mathbf{X}\mathbf{Y}^T] - E[\mathbf{X}]E[\mathbf{Y}]^T \end{aligned} \tag{25}$$

$$\mathrm{C}_{\mathbf{YX}} = \mathrm{C}_{\mathbf{XY}}^T \tag{26}$$

4.2 Bi-variate & Multivariate Normal Distribution

Joint Gaussian distribution of X_1 and X_2 with expectation $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ and covariance matrix $\mathrm{C}_{\mathbf{X}} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ is

$$f_{X_1X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right]\right) \tag{27}$$

Multivariate Gaussian distribution of $\mathbf{X} = (X_1, X_2, \ldots, X_N)^T$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{N/2} \det\left[\mathrm{C}_{\mathbf{X}}\right]} \exp\left\{-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu}\right)^T \mathrm{C}_{\mathbf{X}}^{-1}\left(\mathbf{x}-\boldsymbol{\mu}\right)\right\}, \tag{28}$$

Properties:

- Random vector $\mathbf{X}$ is **jointly** Gaussian distributed, iff (if and only if) for all possible real vectors $\mathbf{a} = (a_1, \ldots, a_n)^T$ linear combination $Y = \mathbf{a}^T\mathbf{X}$ is Gaussian,
$$Y \sim N(\mathbf{a}^T\boldsymbol{\mu}, \mathbf{a}^T C_{\mathbf{X}}\mathbf{a}). \tag{29}$$
- If $X_1, X_2, \ldots, X_N,\ X_k \sim N(0,1), 1 \le k \le n$ are identically and independently distributed (IID) normal Gaussian random variables, it is termed as *normalized Gaussian random vector*. Its joint PDF is given by
$$f_{\mathbf{X}}(\mathbf{x}) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_N}(x_N) = \frac{1}{(2\pi)^{N/2}} \exp\left(-\frac{x_1^2 + x_2^2 + \ldots + x_N^2}{2}\right) = \frac{1}{(2\pi)^{N/2}} \exp\left(-\frac{\mathbf{x}^T\mathbf{x}}{2}\right) \tag{30}$$
The covariance matrix of such vector is given by identity matrix of size $N \times N, C_{\mathbf{X}} = I_n$ and its expectation is $\boldsymbol{\mu} = \mathbf{0}_{N\times 1}$.
- Linear combination of **independent** Gaussian variables, $X_i \sim N(\mu_i, \sigma_i^2)$ is Gaussian
$$\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n}(a_i\sigma_i)^2\right). \tag{31}$$
- Linear transformation – follows Eqs. (23a).
- If jointly distributed Gaussian random variables are *uncorrelated*, they are also *independent*

5 Random Processes – General Properties

- PDF & CDF
$$F_{\mathbf{x}}(x;t) = p(\mathbf{x}(t) \le x) \tag{32a}$$
$$f_{\mathbf{x}}(x;t) = \frac{\partial}{\partial x} F_{\mathbf{x}}(x;t) \tag{32b}$$
$$p_{\mathbf{x}}[x_k;n] = p(\mathbf{x}[n] = x_k) \tag{32c}$$
- Average:
$$E[\mathbf{x}(t)] = \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x;t)dx \tag{33a}$$
$$E\big[\mathbf{x}[n]\big] = \sum_i x_i p_{\mathbf{x}}[x_k;n] \tag{33b}$$
- Variance:
$$\mathrm{Var}\left[\mathbf{x}(t)\right] = E\left[\mathbf{x}^2(t)\right] - E^2\left[\mathbf{x}(t)\right] = \sigma_{\mathbf{x}}(t) \tag{34a}$$
$$\mathrm{Var}\left[\mathbf{x}[n]\right] = E\left[\mathbf{x}^2[n]\right] - E^2\left[\mathbf{x}[n]\right] = \sigma_{\mathbf{x}}[n] \tag{34b}$$
- Auto-correlation
$$R_{\mathbf{x}}(t_1,t_2) = E[\mathbf{x}(t_1)\mathbf{x}(t_2)] \tag{35a}$$
$$R_{\mathbf{x}}(t,t+\tau) = E[\mathbf{x}(t)\mathbf{x}(t+\tau)] \tag{35b}$$
$$R_{\mathbf{x}}(t_1,t_2) = R_{\mathbf{x}}(t_2,t_1) \tag{35c}$$
$$R_{\mathbf{x}}(t,t) = E[\mathbf{x}^2(t)] \tag{35d}$$
$$R_{\mathbf{x}}[n_1,n_2] = E\left[\mathbf{x}[n_1]\mathbf{x}[n_2]\right] \tag{35e}$$
$$R_{\mathbf{x}}[n,n] = E\left[\mathbf{x}^2[n]\right] \tag{35f}$$
- Auto-covariance
$$C_{\mathbf{x}}(t_1,t_2) = E\left[\left\{\mathbf{x}(t_1) - E[\mathbf{x}(t_1)]\right\}\left\{\mathbf{x}(t_2) - E[\mathbf{x}(t_2)]\right\}\right] \tag{36}$$
$$= R_{\mathbf{x}}(t_1,t_2) - E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)] \tag{37}$$
$$C_{\mathbf{x}}[n_1,n_2] = E\left[\left\{\mathbf{x}[n_1] - E[\mathbf{x}[n_1]]\right\}\left\{\mathbf{x}[n_2] - E[\mathbf{x}[n_2]]\right\}\right] \tag{38}$$
$$= R_{\mathbf{x}}[n_1,n_2] - E\left[\mathbf{x}[n_1]\right]E\left[\mathbf{x}[n_2]\right] \tag{39}$$
$$C_{\mathbf{x}}(t,t) = Var[\mathbf{x}(t)] \tag{40a}$$
$$C_{\mathbf{x}}[n,n] = Var[\mathbf{x}[n]] \tag{40b}$$
- Correlation Coefficient
$$\rho_{\mathbf{x}}(t_1,t_2) = \frac{C_{\mathbf{x}}(t_1,t_2)}{\sqrt{C_{\mathbf{x}}(t_1,t_1)C_{\mathbf{x}}(t_2,t_2)}} \tag{41a}$$
$$\left|\rho_{\mathbf{x}}(t_1,t_2)\right| \leqslant 1 \tag{41b}$$
- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *orthogonal*, $R_{\mathbf{x}}(t_1,t_2) = 0$.
- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *uncorrelated*, $C_{\mathbf{x}}(t_1,t_2) = \rho_{\mathbf{x}}(t_1,t_2) = 0$.
- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *independent*, $R_{\mathbf{x}}(t_1,t_2) = E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)]$.

6 Wide-Sense Stationary (WSS) Process

Definition:

$$E[\mathbf{x}(t)] = E[\mathbf{x}(0)] = \mu_{\mathbf{x}} = \text{const} \tag{42a}$$
$$R_{\mathbf{x}}(t_1, t_2) = R_{\mathbf{x}}(\tau = |t_2 - t_1|), \quad \forall t_1, t_2 \tag{42b}$$
$$E[\mathbf{x}[n]] = E[\mathbf{x}[0]] = \mu_{\mathbf{x}} = \text{const} \tag{42c}$$
$$R_{\mathbf{x}}[n_1, n_2] = R_{\mathbf{x}}(k = |n_2 - n_1|), \quad \forall n_1, n_2 \tag{42d}$$

- Auto-correlation

$$R_{\mathbf{x}}(\tau) = E[\mathbf{x}(t)\mathbf{x}(t+\tau)] \tag{43a}$$
$$R_{\mathbf{x}}[k] = E\left[\mathbf{x}[n]\mathbf{x}[n+k]\right] \tag{43b}$$

Properties:

$$R_{\mathbf{x}}(-\tau) = R_{\mathbf{x}}(\tau) \tag{44a}$$
$$R_{\mathbf{x}}(0) = E[\left|\mathbf{x}(0)\right|^2] = E[\left|\mathbf{x}(t)\right|^2] \tag{44b}$$
$$Var[\mathbf{x}(t)] = C_{\mathbf{x}}(0) = \sigma_{\mathbf{x}}^2 \tag{44c}$$
$$R_{\mathbf{x}}(0) \geqslant \left|R_{\mathbf{x}}(\tau)\right| \tag{44d}$$

- Auto-covariance

$$C_{\mathbf{x}}(\tau) = R_{\mathbf{x}}(\tau) - \mu_{\mathbf{x}}^2 \tag{45a}$$
$$C_{\mathbf{x}}[k] = R_{\mathbf{x}}[k] - \mu_{\mathbf{x}}^2 \tag{45b}$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(\tau) = \frac{C_{\mathbf{x}}(\tau)}{C_{\mathbf{x}}(0)} \tag{46a}$$
$$\rho_{\mathbf{x}}[k] = \frac{C_{\mathbf{x}}[k]}{C_{\mathbf{x}}[0]} \tag{46b}$$

6.1 Power Spectral Density (PSD)

$$S_{\mathbf{x}}(F) = \mathscr{F}\left\{R_{\mathbf{x}}(\tau)\right\} = \qquad -\infty \leq f \leq \infty$$
$$= \int_{-\infty}^{\infty} R_{\mathbf{x}}(\tau)\exp\left(-j2\pi F\tau\right) d\tau \tag{47a}$$
$$R_{\mathbf{x}}(\tau) = \mathscr{F}^{-1}\left\{S_{\mathbf{x}}(F)\right\} =$$
$$= \int_{-\infty}^{\infty} S_{\mathbf{x}}(f)\exp\left(j2\pi F\tau\right) dF \tag{47b}$$
$$S_{\mathbf{x}}(f) = \text{DTFT}\left\{R_{\mathbf{x}}[k]\right\} = \sum_{k=-\infty}^{\infty} R_{\mathbf{x}}[k]e^{-j2\pi fk} \tag{47c}$$

Properties:

$$S_{\mathbf{x}}(F) = S_{\mathbf{x}}(-F) \tag{48a}$$
$$S_{\mathbf{x}}(F) \geqslant 0,\ \forall F \tag{48b}$$
$$S_{\mathbf{x}}(F) \in \mathbb{R} \tag{48c}$$
$$S_{\mathbf{x}}(f) = S_{\mathbf{x}}(-f) \tag{48d}$$
$$S_{\mathbf{x}}(f) \geqslant 0,\ \forall f \tag{48e}$$
$$S_{\mathbf{x}}(f) \in \mathbb{R} \tag{48f}$$
$$S_{\mathbf{x}}(f) = S_{\mathbf{x}}(f+1) \tag{48g}$$

Average power

$$P_{\mathbf{x}} = E\left[\mathbf{x}^2(t)\right] = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(F)dF \tag{49a}$$
$$P_{\mathbf{x}} = E\left[\mathbf{x}^2[n]\right] = R_{\mathbf{x}}[0] = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{\mathbf{x}}(f)df \tag{49b}$$

6.2 White Noise & White Gaussian Noise (WGN) Process

White noise process is SSS (WSS) process that is characterized by

$$R_{\mathbf{n}}(\tau) = \sigma^2\delta(\tau) \tag{50a}$$
$$S_{\mathbf{n}}(F) = \sigma^2 \quad \forall F \tag{50b}$$

For WGN process, $\mathbf{n}(t) \sim N(0, \sigma^2)$,

$$R_{\mathbf{n}}(\tau) = \frac{N_0}{2}\delta(\tau) \tag{51a}$$
$$S_{\mathbf{n}}(F) = \frac{N_0}{2} \quad \forall F \tag{51b}$$

6.3 Relation Between Covariance Matrix & Auto-covariance

Given WSS process $\mathbf{x}(t)$, the corresponding correlation matrix of $\mathbf{X} = [\mathbf{x}(t_1), \dots, \mathbf{x}(t_N)]^T$ is given by

$$R_{\mathbf{X}} = E\left[\mathbf{X}\mathbf{X}^T\right] \tag{52}$$
$$R_{\mathbf{X}}(i,j) = E\left[X_i X_j\right] = R_{\mathbf{x}}\left(|t_i - t_j|\right) \tag{53}$$

7 Cross-Signal

- Cross-correlation

$$R_{\mathbf{xy}}(t_1, t_2) = E[\mathbf{x}(t_1)\mathbf{y}(t_2)] \tag{54}$$

- Cross-covariance

$$C_{\mathbf{xy}}(t_1, t_2) = R_{\mathbf{xy}}(t_1, t_2) - E[\mathbf{x}(t_1)]E[\mathbf{y}(t_2)] \tag{55}$$

- Correlation Coefficient

$$\rho_{\mathbf{xy}}(t_1, t_2) = \frac{C_{\mathbf{xy}}(t_1, t_2)}{\sqrt{C_{\mathbf{x}}(t_1, t_1)C_{\mathbf{y}}(t_2, t_2)}} \tag{56}$$

7.1 WSS Cross-signal

- $\mathbf{x}(t), \mathbf{y}(t)$ are jointly WSS, if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ each of them is WSS and

$$R_{\mathbf{xy}}(\tau) = E[\mathbf{x}(t)\mathbf{y}(t+\tau)] \tag{57}$$

- When $\mathbf{x}(t)$ and $\mathbf{y}(t+\tau)$ are *uncorrelated jointly WSS*, $C_{\mathbf{xy}}(\tau) = 0$.

  Properties

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{yx}}(-\tau) \tag{58a}$$
$$\left|R_{\mathbf{xy}}(\tau)\right| \leqslant \sqrt{R_{\mathbf{x}}(0)R_{\mathbf{y}}(0)} \tag{58b}$$
$$\left|R_{\mathbf{xy}}(\tau)\right| \leqslant \frac{1}{2}\left[R_{\mathbf{x}}(0) + R_{\mathbf{y}}(0)\right] \tag{58c}$$

- Cross-covariance

$$C_{\mathbf{xy}}(\tau) = R_{\mathbf{xy}}(\tau) - \mu_{\mathbf{x}}\mu_{\mathbf{y}} \tag{59}$$

- Cross-PSD

$$S_{\mathbf{xy}}(f) = \mathscr{F}\left\{R_{\mathbf{xy}}(\tau)\right\} \tag{60}$$

Properties

$$S_{\mathbf{xy}}(f) = S_{\mathbf{yx}}(-f) = S^*_{\mathbf{xy}}(-f) \tag{61}$$

Correlation coefficient

$$\rho_{\mathbf{xy}}(\tau) = \frac{C_{\mathbf{xy}}(\tau)}{C_{\mathbf{xy}}(0)} \tag{62}$$

- Coherence

$$\gamma_{\mathbf{xy}}(f) = \frac{S_{\mathbf{xy}}(f)}{\sqrt{S_{\mathbf{x}}(f)S_{\mathbf{y}}(f)}} \tag{63}$$

8 LTI and WSS Random Process

Output of LTI system with impulse response $h(t)$ and random process $x(t)$,

$$y(t) = x(t) * h(t) \tag{64}$$

Average

$$m_{\mathbf{y}} = m_{\mathbf{x}} \int_{-\infty}^{\infty} h(s)ds = m_{\mathbf{x}} H(f=0) \tag{65}$$

Cross-correlation & cross-covariance:

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) \tag{66a}$$
$$C_{\mathbf{xy}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) \tag{66b}$$
$$R_{\mathbf{yx}}(\tau) = R_{\mathbf{x}}(\tau) * h(-\tau) \tag{66c}$$
$$C_{\mathbf{yx}}(\tau) = C_{\mathbf{x}}(\tau) * h(-\tau) \tag{66d}$$
$$R_{\mathbf{y}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \tag{66e}$$
$$C_{\mathbf{y}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \tag{66f}$$

Power-Spectral Density (PSD) & Cross-PSD: Given frequency response $H(F) = \mathscr{F}\left\{h(\tau)\right\}$, $H^*(F) = \mathscr{F}\left\{h(-\tau)\right\}$

$$S_{\mathbf{xy}}(F) = S_{\mathbf{x}}(F)H(F) \tag{67a}$$
$$S_{\mathbf{yx}}(F) = S_{\mathbf{x}}(F)H^*(F) \tag{67b}$$
$$S_{\mathbf{y}}(F) = S_{\mathbf{x}}(F)H(F)H^*(F) = S_{\mathbf{x}}(f)\left|H(F)\right|^2 \tag{67c}$$

Power of the process:

$$P_x = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(F)\,dF \tag{68a}$$
$$P_y = R_{\mathbf{y}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(F)\left|H(F)\right|^2 dF \tag{68b}$$
$$P_x = R_{\mathbf{x}}[0] = \int_{-1/2}^{1/2} S_{\mathbf{x}}(f)\,df \tag{68c}$$
$$P_y = R_{\mathbf{y}}[0] = \int_{-1/2}^{1/2} S_{\mathbf{x}}(f)\left|H(f)\right|^2 df \tag{68d}$$

Same process passes two different systems

$$R_{\mathbf{yz}}(\tau) = R_{\mathbf{x}}(\tau) * h_1(-\tau) * h_2(\tau) \tag{69}$$
$$S_{\mathbf{yz}}(F) = S_{\mathbf{x}}(F)H_1^*(F)H_2(F) \tag{70}$$

8.1 Z-Transform

Auto-correlation

$$H(z) = \mathscr{Z}\left\{h[n]\right\} = \frac{B(z)}{A(z)}$$
$$\mathscr{Z}\left\{h[n] * h[-n]\right\} = \frac{B(z)B(z^{-1})}{A(z)A(z^{-1})}$$
$$S_{\mathbf{x}}(z) = \mathscr{Z}\left\{R_{\mathbf{x}}[n]\right\}$$

PSD

$$S_{\mathbf{xy}}(z) = S_{\mathbf{x}}(z)H(z) \tag{71a}$$
$$S_{\mathbf{yx}}(z) = S_{\mathbf{x}}(z)H(z^{-1}) \tag{71b}$$
$$S_{\mathbf{y}}(z) = S_{\mathbf{x}}(z)H(z)H(z^{-1}) \tag{71c}$$

Two different systems

$$R_{\mathbf{yz}}[k] = R_{\mathbf{x}}[k] * h_1[-k] * h_2[k] \tag{72a}$$
$$S_{\mathbf{yz}}(f) = S_{\mathbf{x}}(f)H_1^*(f)H_2(f) \tag{72b}$$
$$S_{\mathbf{yz}}(z) = S_{\mathbf{x}}(z)H_1(1/z)H_2(z) \tag{72c}$$

8.2 Gaussian Process

A Gaussian process $\mathbf{x}(t)$ a random process that for $\forall k > 0$ and for all times $t_1,\dots,t_k$, the set of random variable $\mathbf{x}(t_1),\dots,\mathbf{x}(t_k)$ is jointly Gaussian (i.e. described by Eq. (28)).

Properties:

- WSS Gaussian process is SSS.
- Gaussian process $\mathbf{x}(t)$ that passes through LTI system, $\mathbf{y}(t) = h(t) * \mathbf{x}(t)$, is also Gaussian process that may be described by the change of expectation and auto-correlation,

$$E[\mathbf{y}(t)] = E[\mathbf{x}(t)] \int_{-\infty}^{\infty} h(s)ds \tag{73a}$$
$$= E[\mathbf{x}(t)]H(0), \quad H(F) = \mathscr{F}\left\{h(t)\right\}$$
$$C_{\mathbf{y}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \tag{73b}$$

- The resulting autocorrelation may be used for producing the correspondent covariance matrix $C_{\mathbf{Y}}$ of a multivariate Gaussian $\mathbf{Y} = [\mathbf{y}(t_1),\dots,\mathbf{y}(t_N)]^T$

8.3 Linear Prediction

Given N samples of process $\mathbf{x}[n]$, and predictor

$$\hat{\mathbf{x}}[n+1] = \sum_{i=1}^{N} a_i \mathbf{x}[n-i+1], \tag{74}$$

the values of a_i are given by a solution of

$$\begin{bmatrix} R_{\mathbf{x}}[0] & R_{\mathbf{x}}[1] & \cdots & R_{\mathbf{x}}[N-1] \\ R_{\mathbf{x}}[1] & R_{\mathbf{x}}[0] & \cdots & R_{\mathbf{x}}[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ R_{\mathbf{x}}[N-1] & R_{\mathbf{x}}[N-2] & \cdots & R_{\mathbf{x}}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} R_{\mathbf{x}}[1] \\ R_{\mathbf{x}}[2] \\ \vdots \\ R_{\mathbf{x}}[N] \end{bmatrix} \tag{75}$$

and the resulting minimum MSE is

$$mse_{min} = R_{\mathbf{x}}[0] - \sum_{i=1}^{N} a_i R_{\mathbf{x}}[i] \tag{76}$$

9 Different Supplementary Formulas

9.1 Derivatives

$$\frac{d}{dx} x^n = n x^{n-1}$$
$$\frac{d}{dx} \exp\left[f(x)\right] = \exp\left[f(x)\right] \frac{d}{dx} f(x)$$

9.2 Integrals

9.2.1 Indefinite

$$\int x^n dx = \frac{1}{n+1} x^{n+1},\ n \neq -1$$
$$\int \exp(ax) dx = \frac{1}{a} \exp(ax)$$
$$\int x \exp(ax) dx = \exp(ax) \left[\frac{x}{a} - \frac{1}{a^2}\right]$$
$$\int x^2 \exp(ax) dx = \exp(ax) \left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right]$$
$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

9.2.2 Definite

$$\int_0^\infty \exp\left(-a^2 x^2\right) dx = \frac{\sqrt{\pi}}{2a}$$
$$\int_0^\infty x^2 \exp\left(-a^2 x^2\right) dx = \frac{\sqrt{\pi}}{4a^3}$$
$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$
$$\int_{-\infty}^{\infty} f(x) \delta(x-a) dx = f(a)$$

9.3 Fourier Transform

9.3.1 Properties

$$\frac{d^n}{dt^n} f(t) \overset{\mathscr{F}}{\Longleftrightarrow} (j2\pi f)^n F(f)$$
$$f(-t) \overset{\mathscr{F}}{\Longleftrightarrow} F^*(f)$$
$$f(t-t_0) \overset{\mathscr{F}}{\Longleftrightarrow} F(f) e^{-j2\pi f t_0}$$
$$f(t) e^{j2\pi f_0 t} \overset{\mathscr{F}}{\Longleftrightarrow} F(f-f_0)$$

9.3.2 Transform pairs

$$u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2}\left(\frac{1}{j\pi f} + \delta(f)\right)$$
$$\exp(-at) u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{a + j2\pi f}$$
$$t \exp(-at) u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{(a + j2\pi f)^2}$$
$$\exp\left(-a|t|\right) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{2a}{a^2 + 4\pi^2 f^2}$$
$$\exp\left(-at^2\right) \overset{\mathscr{F}}{\Longleftrightarrow} \sqrt{\frac{\pi}{a}} \exp\left(-\frac{(\pi f)^2}{a}\right)$$
$$\cos\left(2\pi f_a t\right) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2}\left[\delta(f - f_a) + \delta(f + f_a)\right]$$
$$\sin\left(2\pi f_a t\right) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2j}\left[\delta(f - f_a) - \delta(f + f_a)\right]$$

9.4 Convolution

$$x(t) * y(t) = \int_{-\infty}^{\infty} f(s) g(t-s) ds$$
$$x(t) * y(t) \overset{\mathscr{F}}{\Longleftrightarrow} X(f) Y(f)$$
$$\delta(t) * y(t) = y(t)$$

9.5 Trigonometry

$$\sin^2(\alpha) = \frac{1}{2}\Big(1-\cos(2\alpha)\Big)$$
$$\cos^2(\alpha) = \frac{1}{2}\Big(1+\cos(2\alpha)\Big)$$
$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\Big[\cos(\alpha+\beta)+\cos(\alpha-\beta)\Big]$$
$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\Big[\cos(\alpha-\beta)-\cos(\alpha+\beta)\Big]$$
$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\Big[\sin(\alpha-\beta)+\sin\big((\alpha+\beta)\big)\Big]$$

9.6 Matrices

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
$$\det[\mathbf{A}] = ad - bc$$
$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

10 Z-transforms

$$X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$

10.1 Usual Transforms

Signal	Z transform	ROC
$\delta[n]$	1	$\mathbb{C}$
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	$\mathbb{C}-\{0\}$ if $m>0$, $\mathbb{C}-\{\infty\}$ if $m<0$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < a$

10.2 Properties

Property	Discrete Signal	Z transform	ROC
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(z) + a_2 X_2(z)$	includes $R_1 \cap R_2$
Time shift	$x[n-n_0]$	$z^{-n_0} X(z)$	R
Frequency scaling	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$\lvert z_0\rvert R$
Time reversal	$x[-n]$	$X(z^{-1})$	R^{-1} if $m<0$
Convolution	$(x_1 * x_2)[n]$	$X_1(z) X_2(z)$	$R_1 \cap R_2$ (or possibly more)
Time differentiation	$x[n] - x[n-1]$	$(1-z^{-1})X(z)$	$R \cap \{\lvert z\rvert > 0\}$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{X(z)}{1-z^{-1}}$	$R \cap \{\lvert z\rvert > 1\}$