המחלקה להנדסת חשמל ואלקטרוניקה

תאריך הבחינה: 27.02.19
שעות הבחינה: 09:00-12:00

מבוא לאותות אקראיים

מועד ב'

ד"ר דימה בחובסקי, אלן פריד

תשע"ט סמסטר א'

חומר עזר - דף נוסחאות אישי (משני צדדים), מחשבון

הוראות מיוחדות:

- סעיפים הם בעלי ניקוד זהה, אלא אם צוין אחרת.
- יש לציין באופן מלא וברור את שלבי הפתרון. תשובה ללא הסבר לא תתקבלנה.
- במקום בו נדרש חישוב מספרי, יש קודם לרשום את הנוסחא, ורק אח"כ להציב!
- יש לציין יחידות למספרים, ובמידה וקיימות!
- כל השרטוטים יהיו גדולים, ברורים, עם סימון צירים!
- אין חובה להגיע לערך מספרי של הפונקציה $Q(x)$, במידה ומופיעה בתשובה.

השאלון כולל 11 דפים (כולל דף זה)

בהצלחה!

1 שאלה (108 נק')

הבהרה: אין חובה להגיע לערך מספרי של הפונקציה $Q(x)$, במידה ומופיעה בתשובה.

נתון תהליך $\mathbf{x}[n]$, בעל תכונות הבאות: סטציאונרי, גאוסי,

$$E\big[\mathbf{x}[n]\big] = 0$$
$$R_{\mathbf{x}}[k] = 4\exp\big(-|k|\big)$$

נתון תהליך אקראי

$$\mathbf{y}[n] = \mathbf{x}[n] + 2\mathbf{x}[n-2]$$

(א) חשב הסתברות עבור $p\big(\mathbf{x}[n] < 4\big)$.

(ב) מהו ערך מספרי של מקדם קורלציה בין משתנים אקראיים $\mathbf{x}[1], \mathbf{x}[3]$?

(ג) מעוניינים לבצע טרנספורמציה לינארית (הכפלה במטריצה) על הערכים של וקטור $\big[\mathbf{x}[1], \mathbf{x}[3]\big]^T$, על מנת לקבל וקטור חדש $\mathbf{z} = \big[z_1, z_2\big]^T$ בעל משתנים אקראיים חסרי קורלציה. מהי מטריצת covariance של $\mathbf{z}$ המתקבל (ערכים מספריים)? אין צורך לחשב מטריצת טרנספורמציה.

(ד) הוכח, ש-$\mathbf{y}[n]$ הינו תהליך WSS.

(ה) חשב הסתברות עבור $p\big(\mathbf{y}[n] > 4\big)$.

(ו) מהי מטריצת covariance בין משתנים אקראיים $\mathbf{y}[1], \mathbf{y}[3]$? ניתן להשאיר את התשובה כפונ' של $R_{\mathbf{x}}[k]$ בלי להגיע לערך מספרי.

(ז) האם תהליכים $\mathbf{x}[n], \mathbf{y}[n]$ הם סטציאונריים במשותף?

(ח) (20 נק') מעוניינים לעשות חיזוי לינארי של $\mathbf{x}[n]$ מתוך $\mathbf{y}[n]$. עבור חיזוי מהצורה

$$\hat{\mathbf{x}}[n+1] = a_0\mathbf{y}[n] + a_1\mathbf{y}[n-1]$$

חשב מספרית את הערכים של a_0, a_1.

(ט) נתון תהליך אקראי $\mathbf{w}[n] = \mathbf{x}^2[n]$. חשב $E\big[\mathbf{w}[n]\big]$.

Random Processes – Formulas

1 Distributions

1.1 Continuous

	Notation	PDF	CDF	$E[X]$	Var$[X]$
Uniform	U[a,b]	$\begin{cases} \frac{1}{b-a} & a \leqslant x \leqslant b \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leqslant x \leqslant b \\ 1 & b < x \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal	$N(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$	$\Phi(x)$	μ	σ^2
Exponential	$Exp(\lambda)$	$\lambda \exp(-\lambda x), x \geqslant 0$	$1 - \exp(-\lambda x)$	$1/\lambda$	$1/\lambda^2$

1.1.1 Q-function

Given $Y \sim N(\mu, \sigma^2)$

$$p(Y > y) = Q\left(\frac{y-\mu}{\sigma}\right) \tag{1a}$$

$$Q(x) = 1 - \Phi(x) \tag{1b}$$

$$Q(-x) = 1 - Q(x) \tag{1c}$$

1.2 Discrete

	Notation	PMF	CDF	$E[X]$	Var$[X]$
Bernoulli	Ber(p)	$\begin{cases} 1-p & k=0 \\ p & k=1 \end{cases}$	$\begin{cases} 0 & x < 0 \\ 1-p & 0 \leqslant x < 1 \\ 1 & 1 \leqslant x \end{cases}$	p	$p(1-p)$
Poisson	$\mathcal{P}(\lambda)$	$p(X=k) = \exp(-\lambda)\frac{\lambda^k}{k!}$	$\exp(-\lambda)\sum_{i=0}^{k}\frac{\lambda^i}{i!}$	λ	λ
Erlang	$Erlang(k, \lambda t)$	$\lambda\frac{(\lambda t)^{k-1}}{(k-1)!}\exp(-\lambda t)$	$1 - \exp(-\lambda t)\sum_{n=0}^{k-1}\frac{(\lambda t)^n}{n!}$	$\frac{k}{\lambda t}$	$\frac{k}{(\lambda t)^2}$

2 Random Variables

Definitions:

$$F_X(x) = p(X \leqslant x) \tag{2a}$$

$$f_X(x) = \frac{\partial F_X(x)}{\partial x} \tag{2b}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(p)\, dp \tag{2c}$$

$$p(a < X \leqslant b) = F_X(b) - F_X(a) \tag{2d}$$

$$p_X[x_k] = p(X = x_k) \tag{3a}$$

$$F_X(x) = \sum_{k: x_k \leqslant x} p_X[x_k] \tag{3b}$$

Expectation:

$$E[X] = \begin{cases} \int_{-\infty}^{\infty} x f_X(x) dx \\ \sum_i x_i p_x[x_i] \end{cases} \tag{4a}$$

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x) dx \\ \sum_i g(x_i) p_x[x_i] \end{cases} \tag{4b}$$

$$E[aX + b] = aE[x] + b \tag{4c}$$

Variance:

$$\begin{aligned} \text{Var}[X] =& E\big[(X - E[X])^2\big] \\ =& E[X^2] - E^2[X] \end{aligned} \tag{5a}$$

$$\text{Var}[aX + b] = a^2 \text{Var}[X] \tag{5b}$$

$$\text{Var}[b] = 0 \tag{5c}$$

2.1 Simulation

Given CDF of the required random variable, $F_X(x)$, it may be generated from $Z \sim U(0,1)$ by $X = F_X^{-1}(Z)$. Example: Given $X \sim \text{U}[0,1]$, $Y = a + (b-a)X \sim \text{U}[a,b]$

3 Two Random Variables

3.1 Joint Distributions

Definitions:

$$F_{XY}(x,y) = p(X \leqslant x, Y \leqslant y) \tag{6a}$$

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \geqslant 0 \tag{6b}$$

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(s,p)\, dp\, ds \tag{6c}$$

$$p[x_j, y_k] = p(X = x_j, Y = y_k) \tag{7a}$$

$$F_{XY}(x,y) = p(X \leqslant x_j, Y \leqslant y_k) \tag{7b}$$

Expectation:

$$E[XY] = \iint xy f_{XY}(x,y) dx dy \tag{8a}$$

$$E[g(X)] = \begin{cases} \iint g(x,y) f_{XY}(x,y) dx dy \\ \sum_i \sum_k g(x_i, y_k) p_x[x_i, y_k] \end{cases} \tag{8b}$$

$$E[aX + bY] = aE[X] + bE[Y] \tag{8c}$$

For **independent** random variables:

$$f_{XY}(x,y) = f_X(x) f_Y(y) \tag{9a}$$

$$p_{XY}[x_k, y_j] = p_X[x_k] p_Y[y_j] \tag{9b}$$

$$F_{XY}(x,y) = F_X(x) F_Y(y) \tag{9c}$$

$$E[XY] = E[X]E[Y] \tag{9d}$$

$$E[g_1(X) g_2(Y)] = E[g_1(X)] E[g_2(Y)] \tag{9e}$$

$$\text{Var}[aX + bY] = a^2 \text{Var}[X] + b^2 \text{Var}[Y] \tag{9f}$$

Marginal distribution:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) dy \tag{10a}$$

$$p_X[x_k] = \sum_j p_{XY}[x_k, y_j] \tag{10b}$$

$$F_X(x) = F_{XY}(x, \infty) \tag{10c}$$

$$F_Y(y) = F_{XY}(\infty, y) \tag{10d}$$

3.2 Conditional Relations

Conditional distribution (Bayes),
for $f_X(x), f_Y(y), p_X[x_k], p_Y[y_k] > 0$:

$$f_{Y|X}(y|x) f_X(x) = f_{X|Y}(x|y) f_Y(x) = f_{XY}(x,y) \tag{11a}$$

$$p_{Y|X}[y_j|x_k] p_X[x_k] = p_{X|Y}[x_k|y_j] p_Y[y_j] = p_{XY}[x_k, y_j] \tag{11b}$$

$$F_{Y|X}(y|x) = p(Y \leqslant y | X = x) \tag{11c}$$

$$= \int_{-\infty}^{y} f_{Y|X}(s|x) ds \tag{11d}$$

$$F_{Y|X}[y|x_k] = \frac{p[Y \leqslant y_j, X = x_k]}{p_X[x_k]} \tag{11e}$$

Conditional expectation & Variance:

$$E[Y|X] = \begin{cases} \int y f_{Y|X}(y|x) dy \\ \sum_j y_j p[y_j|x_k] \end{cases} \tag{12a}$$

$$E[X] = E\big[E[X|Y]\big] = \iint y f_{Y|X}(y|x) f_X(x) dx dy \tag{12b}$$

$$\text{Var}[Y|X] = E[Y^2|X] - E^2[Y|X] \tag{12c}$$

$$\text{Var}[Y] = \text{Var}\big[E[Y|X]\big] + E\big[\text{Var}[Y|X]\big] \tag{12d}$$

3.3 Correlation, Covariance & Correlation Coefficient

- For two jointly-distributed random variables X and Y, covariance is given by

$$\begin{aligned} \text{Cov}[X,Y] =& E\big[(X - E[X])(Y - E[Y])\big] \\ =& E[XY] - E[X]E[Y]. \end{aligned} \tag{13}$$

Main covariance properties are:

$$\mathrm{Cov}[X,X] = Var[X] \tag{14a}$$
$$\mathrm{Cov}[X,Y] = \mathrm{Cov}[Y,X] \tag{14b}$$
$$\mathrm{Cov}[X,a] = 0 \tag{14c}$$
$$\mathrm{Cov}[aX,bY] = ab\,\mathrm{Cov}[X,Y] \tag{14d}$$
$$\mathrm{Cov}[X,Y] = \mathrm{Cov}[X+a,Y+b] \tag{14e}$$
$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X,Y] \tag{14f}$$
$$|E[XY]| \leqslant \sqrt{E[X^2]E[Y^2]} \text{ Cauchy-Schwatz} \tag{14g}$$

- Correlation coefficient (also termed as Pearson product-moment correlation coefficient) is given by

$$\rho_{XY} = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} \tag{15}$$

such that $|\rho_{XY}| \leqslant 1$.

3.4 MMSE Linear Prediction

Mean square error (MSE) of predictor $\hat{Y}$ is given by

$$mse = E[(Y-\hat{Y})^2] \tag{16}$$

Linear prediction of $\hat{Y} = ax + b$ for $X = x$ is

$$\hat{Y} = E[Y] + \frac{\mathrm{Cov}[X,Y]}{\mathrm{Var}[X]}(x - E[X]) \tag{17}$$

and

$$mse_{min} = E\left[\left(Y - (aX+b)\right)^2\right] = \mathrm{Var}(Y)(1-\rho_{XY}^2) \tag{18}$$

When X, Y are jointly Gaussian, this prediction is optimal among **all** possible predictors

3.5 Relations

- When X and Y are *orthogonal*, $E[XY] = 0$.
- When X and Y are *uncorrelated*, $\mathrm{Cov}[X,Y] = \rho_{XY} = 0$.
- When X and Y are *independent*, they are also uncorrelated (see also Eqs. 9).
- When X and Y are *jointly* Gaussian and uncorrelated $\Rightarrow$ X and Y are independent.
- Joint$\Rightarrow$ marginal, marginal $\not\Rightarrow$ joint

4 Multi-dimensional Random Variables

4.1 Covariance matrix

Given random vector $\mathbf{X} = (X_1, X_2, \ldots, X_N)^T$,

$$\begin{aligned}\mathrm{C}_{\mathbf{X}} &= \mathrm{Cov}[\mathbf{X},\mathbf{X}] = E[(\mathbf{X}-E[\mathbf{X}])(\mathbf{X}-E[\mathbf{X}])^T] \\ &= E[\mathbf{X}\mathbf{X}^T] - E[\mathbf{X}]E[\mathbf{X}]^T \\ &= \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1,X_2] & \cdots & \mathrm{Cov}[X_1,X_N] \\ \mathrm{Cov}[X_2,X_1] & \mathrm{Var}[X_2] & \cdots & \mathrm{Cov}[X_2,X_N] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}[X_N,X_1] & \mathrm{Cov}[X_N,X_2] & \cdots & \mathrm{Var}[X_N] \end{bmatrix}\end{aligned} \tag{19}$$

Properties:

- Symmetry

$$\mathrm{C}_{\mathbf{X}} = \mathrm{C}_{\mathbf{X}}^T \quad \mathrm{Cov}[X_i,X_j] = \mathrm{Cov}[X_j,X_i] \tag{20}$$

- Variance of linear combination: Given vector $\mathbf{a} = (a_1, a_2, \ldots, a_N)^T$,

$$\mathrm{Var}[\mathbf{a}^T\mathbf{X}] = \mathbf{a}^T C_{\mathbf{X}}\mathbf{a} \tag{21}$$

- Linear transformation: Given linear tranformation $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$,

$$E[\mathbf{Y}] = \mathbf{A}E[\mathbf{X}] + \mathbf{b} \tag{22a}$$
$$C_{\mathbf{Y}} = \mathbf{A}C_{\mathbf{X}}\mathbf{A}^T \tag{22b}$$

- Uncorrelated variables

$$\mathrm{C}_{\mathbf{X}} = \mathrm{diag}\left[\mathrm{Var}[X_1], \mathrm{Var}[X_2], \ldots, \mathrm{Var}[X_N]\right] \tag{23}$$

- Cross-covariance: For two random vectors $\mathbf{X} \in \mathbb{R}^m$ and $\mathbf{Y} \in \mathbb{R}^n$, the resulting $m \times n$ cross-covariance matrix is given by

$$\begin{aligned}\mathrm{Cov}[\mathbf{X},\mathbf{Y}] &= C_{\mathbf{XY}} \\ &= E[(\mathbf{X}-E[\mathbf{X}])(\mathbf{Y}-E[\mathbf{Y}])^T] \\ &= E[\mathbf{X}\mathbf{Y}^T] - E[\mathbf{X}]E[\mathbf{Y}]^T\end{aligned} \tag{24}$$
$$\mathrm{C}_{\mathbf{YX}} = \mathrm{C}_{\mathbf{XY}}^T \tag{25}$$

4.2 Decorrelation

Given random vector $\mathbf{X}$ with covariance matrix $\mathrm{C}_{\mathbf{X}}$, and linear transformation $\mathbf{Y} = \mathbf{V}^T\mathbf{X}$, where $\mathbf{V}$ is eigenvectors matrix of $\mathrm{C}_{\mathbf{X}}$, the resulting covariance matrix $\mathrm{C}_{\mathbf{Y}}$ is of the form $\mathrm{C}_{\mathbf{Y}} = \mathrm{diag}\left[\lambda_1, \ldots, \lambda_N\right]$, where λ_i are eigenvalues of $\mathrm{C}_{\mathbf{X}}$.

4.3 Bi-variate & Multivariate Normal Distribution

Joint Gaussian distribution of X_1 and X_2 with expectation $\boldsymbol{\mu} = \begin{bmatrix}\mu_1\\\mu_2\end{bmatrix}$ and covariance matrix $\mathrm{C}_{\mathbf{X}} = \begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\\rho\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}$ is

$$f_{X_1X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}-\frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right]\right) \tag{26}$$

Multivariate Gaussian distribution of $\mathbf{X} = (X_1, X_2, \ldots, X_N)^T$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{N/2}\det\left[\mathrm{C}_{\mathbf{X}}\right]}\exp\left\{-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu}\right)^T\mathrm{C}_{\mathbf{X}}^{-1}\left(\mathbf{x}-\boldsymbol{\mu}\right)\right\}, \tag{27}$$

Properties:

- Random vector $\mathbf{X}$ is **jointly** Gaussian distributed, iff (if and only if) for all possible real vectors $\mathbf{a} = (a_1, \ldots, a_n)^T$ linear combination $Y = \mathbf{a}^T\mathbf{X}$ is Gaussian,

$$Y \sim N(\mathbf{a}^T\boldsymbol{\mu}, \mathbf{a}^T C_{\mathbf{X}}\mathbf{a}). \tag{28}$$

- If $X_1, X_2, \ldots, X_N$, $X_k \sim N(0,1), 1 \leqslant k \leqslant n$ are identically and independently distributed (IID) normal Gaussian random variables, it is termed as *normalized Gaussian random vector*. Its joint PDF is given by

$$f_{\mathbf{X}}(\mathbf{x}) = f_{X_1}(x_1)f_{X_2}(x_2)\cdots f_{X_N}(x_N) = \frac{1}{(2\pi)^{N/2}}\exp\left(-\frac{x_1^2+x_2^2+\ldots+x_N^2}{2}\right) = \frac{1}{(2\pi)^{N/2}}\exp\left(-\frac{\mathbf{x}^T\mathbf{x}}{2}\right) \tag{29}$$

  The covariance matrix of such vector is given by identity matrix of size $N \times N, C_{\mathbf{X}} = I_n$ and its expectation is $\boldsymbol{\mu} = \mathbf{0}_{N\times 1}$.

- Linear combination of **independent** Gaussian variables, $X_i \sim N(\mu_i, \sigma_i^2)$ is Gaussian

$$\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n}(a_i\sigma_i)^2\right). \tag{30}$$

- Linear transformation – follows Eqs. (22a).
- If jointly distributed Gaussian random variables are *uncorrelated*, they are also *independent*

5 Random Processes – General Properties

- PDF & CDF

$$F_{\mathbf{x}}(x;t) = p(\mathbf{x}(t) \leqslant x) \tag{31a}$$
$$f_{\mathbf{x}}(x;t) = \frac{\partial}{\partial x}F_{\mathbf{x}}(x;t) \tag{31b}$$
$$p_{\mathbf{x}}[x_k;n] = p(\mathbf{x}[n] = x_k) \tag{31c}$$

- Average:

$$E[\mathbf{x}(t)] = \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x;t)dx \tag{32a}$$
$$E\big[\mathbf{x}[n]\big] = \sum_i x_i p_{\mathbf{x}}[x_k;n] \tag{32b}$$

- Variance:

$$Var[\mathbf{x}(t)] = E[\mathbf{x}^2(t)] - E^2[\mathbf{x}(t)] \tag{33a}$$
$$Var[\mathbf{x}[n]] = E[\mathbf{x}^2[n]] - E^2[\mathbf{x}[n]] \tag{33b}$$

- Auto-correlation

$$R_{\mathbf{x}}(t_1,t_2) = E[\mathbf{x}(t_1)\mathbf{x}(t_2)] \tag{34a}$$
$$R_{\mathbf{x}}(t,t+\tau) = E[\mathbf{x}(t)\mathbf{x}(t+\tau)] \tag{34b}$$
$$R_{\mathbf{x}}(t_1,t_2) = R_{\mathbf{x}}(t_2,t_1) \tag{34c}$$
$$R_{\mathbf{x}}(t,t) = E[\mathbf{x}^2(t)] \tag{34d}$$
$$R_{\mathbf{x}}[n_1,n_2] = E[\mathbf{x}[n_1]\mathbf{x}[n_2]] \tag{34e}$$

- Auto-covariance

$$\begin{aligned}C_{\mathbf{x}}(t_1,t_2) &= R_{\mathbf{x}}(t_1,t_2) - E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)]\\ &= E[\{\mathbf{x}(t_1) - E[\mathbf{x}(t_1)]\}\{\mathbf{x}(t_2) - E[\mathbf{x}(t_2)]\}]\end{aligned} \tag{35}$$

$$C_{\mathbf{x}}(t,t) = Var[\mathbf{x}(t)] \tag{36}$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(t_1,t_2) = \frac{C_{\mathbf{x}}(t_1,t_2)}{\sqrt{C_{\mathbf{x}}(t_1,t_1)C_{\mathbf{x}}(t_2,t_2)}} \tag{37a}$$

$$|\rho_{\mathbf{x}}(t_1,t_2)| \leqslant 1 \tag{37b}$$

- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *orthogonal*, $R_{\mathbf{x}}(t_1,t_2) = 0$.
- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *uncorrelated*, $C_{\mathbf{x}}(t_1,t_2) = \rho_{\mathbf{x}}(t_1,t_2) = 0$.
- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *independent*, $R_{\mathbf{x}}(t_1,t_2) = E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)]$.

6 Wide-Sense Stationary (WSS) Process

Definition:

$$E[\mathbf{x}(t)] = E[\mathbf{x}(0)] = \mu_{\mathbf{x}} = \text{const} \tag{38a}$$

$$R_{\mathbf{x}}(t_1,t_2) = R_{\mathbf{x}}(\tau = |t_2 - t_1|), \quad \forall t_1,t_2 \tag{38b}$$

$$E[\mathbf{x}[n]] = E[\mathbf{x}[0]] = \mu_{\mathbf{x}} = \text{const} \tag{38c}$$

$$R_{\mathbf{x}}[n_1,n_2] = R_{\mathbf{x}}(k = |n_2 - n_1|), \quad \forall n_1,n_2 \tag{38d}$$

- Auto-correlation

$$R_{\mathbf{x}}(\tau) = E[\mathbf{x}(t)\mathbf{x}(t+\tau)] \tag{39a}$$

$$R_{\mathbf{x}}[k] = E[\mathbf{x}[n]\mathbf{x}(n+k)] \tag{39b}$$

Properties:

$$R_{\mathbf{x}}(-\tau) = R_{\mathbf{x}}(\tau) \tag{40a}$$

$$R_{\mathbf{x}}(0) = E[|\mathbf{x}(0)|^2] = E[|\mathbf{x}(t)|^2] \tag{40b}$$

$$Var[\mathbf{x}(t)] = C_{\mathbf{x}}(0) = \sigma_{\mathbf{x}}^2 \tag{40c}$$

$$R_{\mathbf{x}}(0) \geqslant |R_{\mathbf{x}}(\tau)| \tag{40d}$$

Deterministic definition ($x[n], x(t)$ <u>not</u> random)

$$R_x(\tau) = x(\tau) * x(-\tau) = \int_{-\infty}^{\infty} x(s)x(x+\tau)ds \tag{41a}$$

$$R_x[k] = x[k] * x[-k] = \sum_{n=-\infty}^{\infty} x[n]x[n+k] \tag{41b}$$

- Auto-covariance

$$C_{\mathbf{x}}(\tau) = R_{\mathbf{x}}(\tau) - \mu_{\mathbf{x}}^2 \tag{42a}$$

$$C_{\mathbf{x}}[k] = R_{\mathbf{x}}[k] - \mu_{\mathbf{x}}^2 \tag{42b}$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(\tau) = \frac{C_{\mathbf{x}}(\tau)}{C_{\mathbf{x}}(0)} \tag{43a}$$

$$\rho_{\mathbf{x}}[k] = \frac{C_{\mathbf{x}}[k]}{C_{\mathbf{x}}[0]} \tag{43b}$$

6.1 Linear Prediction

Given N samples of process $\mathbf{x}[n]$, and predictor

$$\hat{\mathbf{x}}[n+1] = \sum_{i=1}^{N} a_i \mathbf{x}[n-i+1], \tag{44}$$

the values of a_i are given by a solution of

$$\begin{bmatrix} R_{\mathbf{x}}[0] & R_{\mathbf{x}}[1] & \cdots & R_{\mathbf{x}}[N-1] \\ R_{\mathbf{x}}[1] & R_{\mathbf{x}}[0] & \cdots & R_{\mathbf{x}}[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ R_{\mathbf{x}}[N-1] & R_{\mathbf{x}}[N-2] & \cdots & R_{\mathbf{x}}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} R_{\mathbf{x}}[1] \\ R_{\mathbf{x}}[2] \\ \vdots \\ R_{\mathbf{x}}[N] \end{bmatrix} \tag{45}$$

and the resulting minimum MSE is

$$mse_{min} = R_{\mathbf{x}}[0] - \sum_{i=1}^{N} a_i R_{\mathbf{x}}[i] \tag{46}$$

6.2 Power Spectral Density (PSD)

$$
\begin{aligned}
S_{\mathbf{x}}(f) &= \mathcal{F}\{R_{\mathbf{x}}(\tau)\} = \qquad -\infty \leqslant f \leqslant \infty \\
&= \int_{-\infty}^{\infty} R_{\mathbf{x}}(\tau)\exp(-j2\pi f\tau)\,d\tau && \text{(47a)}\\
&= 2\int_{0}^{\infty} R_{\mathbf{x}}(\tau)\cos(2\pi f\tau)\,d\tau && \text{(47b)}\\
R_{\mathbf{x}}(\tau) &= \mathcal{F}^{-1}\{S_{\mathbf{x}}(f)\} = \\
&= \int_{-\infty}^{\infty} S_{\mathbf{x}}(f)\exp(j2\pi f\tau)\,df && \text{(47c)}\\
& && \text{(47d)}
\end{aligned}
$$

Properties:

$$
\begin{aligned}
S_{\mathbf{x}}(f) &= S_{\mathbf{x}}(-f) && \text{(48a)}\\
S_{\mathbf{x}}(f) &\geqslant 0,\ \forall f && \text{(48b)}\\
S_{\mathbf{x}}(f) &\in \mathbb{R} \quad \text{(real numbers)} && \text{(48c)}\\
& && \text{(48d)}
\end{aligned}
$$

Average power

$$P_{\mathbf{x}} = E[\mathbf{x}^2(t)] = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(f)df \tag{49}$$

Deterministic definition:

$$S_x(f) = X(f)X^*(f) = |X(f)|^2 \tag{50}$$

6.3 White Noise & White Gaussian Noise (WGN) Process

White noise process is SSS (WSS) process that is characterized by

$$
\begin{aligned}
R_{\mathbf{n}}(\tau) &= \sigma^2\delta(\tau) && \text{(51a)}\\
S_{\mathbf{n}}(f) &= \sigma^2 \quad \forall f && \text{(51b)}
\end{aligned}
$$

For WGN process, $\mathbf{n}(t) \sim N(0, \sigma^2)$,

$$
\begin{aligned}
R_{\mathbf{n}}(\tau) &= \frac{N_0}{2}\delta(\tau) && \text{(52a)}\\
S_{\mathbf{n}}(f) &= \frac{N_0}{2} \quad \forall f && \text{(52b)}
\end{aligned}
$$

6.4 Relation Between Covariance Matrix & Auto-covariance

Given WSS process $\mathbf{x}(t)$, the corresponding correlation matrix of $\mathbf{X} = [\mathbf{x}(t_1), \ldots, \mathbf{x}(t_N)]^T$ is given by

$$
\begin{aligned}
R_{\mathbf{X}} &= E\left[\mathbf{X}\mathbf{X}^T\right] && \text{(53)}\\
R_{\mathbf{X}}(i,j) &= E\left[X_iX_j\right] = R_{\mathbf{x}}\left(|t_i - t_j|\right) && \text{(54)}
\end{aligned}
$$

7 Cross-Signal

- Cross-correlation

$$R_{\mathbf{xy}}(t_1, t_2) = E[\mathbf{x}(t_1)\mathbf{y}(t_2)] \tag{55}$$

- Cross-covariance

$$C_{\mathbf{xy}}(t_1, t_2) = R_{\mathbf{xy}}(t_1, t_2) - E[\mathbf{x}(t_1)]E[\mathbf{y}(t_2)] \tag{56}$$

- Correlation Coefficient

$$\rho_{\mathbf{xy}}(t_1, t_2) = \frac{C_{\mathbf{xy}}(t_1, t_2)}{\sqrt{C_{\mathbf{x}}(t_1, t_1)C_{\mathbf{y}}(t_2, t_2)}} \tag{57}$$

7.1 WSS Cross-signal

- $\mathbf{x}(t), \mathbf{y}(t)$ are jointly WSS, if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ each of them is WSS and

$$R_{\mathbf{xy}}(\tau) = E[\mathbf{x}(t)\mathbf{y}(t+\tau)] \tag{58}$$

- When $\mathbf{x}(t)$ and $\mathbf{y}(t+\tau)$ are *uncorrelated jointly WSS*, $C_{\mathbf{xy}}(\tau) = 0$.

  Properties

$$
\begin{aligned}
R_{\mathbf{xy}}(\tau) &= R_{\mathbf{yx}}(-\tau) && \text{(59a)}\\
|R_{\mathbf{xy}}(\tau)| &\leqslant \sqrt{R_{\mathbf{x}}(0)R_{\mathbf{y}}(0)} && \text{(59b)}\\
|R_{\mathbf{xy}}(\tau)| &\leqslant \frac{1}{2}\left[R_{\mathbf{x}}(0) + R_{\mathbf{y}}(0)\right] && \text{(59c)}
\end{aligned}
$$

  Deterministic definition

$$R_{xy}(\tau) = x(\tau) * y(-\tau) \tag{60}$$

- Cross-covariance

$$C_{\mathbf{xy}}(\tau) = R_{\mathbf{xy}}(\tau) - \mu_{\mathbf{x}}\mu_{\mathbf{y}} \tag{61}$$

- Cross-PSD

$$S_{\mathbf{xy}}(f) = \mathcal{F}\{R_{\mathbf{xy}}(\tau)\} \tag{62}$$

  Properties

$$S_{\mathbf{xy}}(f) = S_{\mathbf{yx}}(-f) = S^*_{\mathbf{xy}}(-f) \tag{63}$$

  Deterministic definition

$$S_{xy}(f) = X(f)Y^*(f) \tag{64}$$

- Coherence

$$\gamma_{\mathbf{xy}}(f) = \frac{S_{\mathbf{xy}}(f)}{\sqrt{S_{\mathbf{x}}(f)S_{\mathbf{y}}(f)}} \tag{65}$$

8 LTI and WSS Random Process

Output of LTI system with impulse response $h(t)$ and random process $x(t)$,

$$y(t) = x(t) * h(t) \tag{66}$$

Average

$$m_{\mathbf{y}} = m_{\mathbf{x}} \int_{-\infty}^{\infty} h(s)ds = m_{\mathbf{x}} H(f=0) \tag{67}$$

Cross-correlation & cross-covariance:

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) \tag{68a}$$
$$C_{\mathbf{xy}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) \tag{68b}$$
$$R_{\mathbf{yx}}(\tau) = R_{\mathbf{x}}(\tau) * h(-\tau) \tag{68c}$$
$$C_{\mathbf{yx}}(\tau) = C_{\mathbf{x}}(\tau) * h(-\tau) \tag{68d}$$
$$R_{\mathbf{y}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \tag{68e}$$
$$C_{\mathbf{y}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \tag{68f}$$

Power-Spectral Density (PSD) & Cross-PSD: Given frequency response $H(f) = \mathcal{F}\{h(\tau)\}$, $H^*(f) = \mathcal{F}\{h(-\tau)\}$

$$S_{\mathbf{xy}}(f) = S_{\mathbf{x}}(f) H(f) \tag{69a}$$
$$S_{\mathbf{yx}}(f) = S_{\mathbf{x}}(f) H^*(f) \tag{69b}$$
$$S_{\mathbf{y}}(f) = S_{\mathbf{x}}(f) H(f)H^*(f) = S_{\mathbf{x}}(f) |H(f)|^2 \tag{69c}$$

Power of the process:

$$P_x = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(f) df \tag{70a}$$

$$P_y = R_{\mathbf{y}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(f) |H(f)|^2 df \tag{70b}$$

8.1 SNR

Given input signal

$$\mathbf{x}(t) = \mathbf{s}(t) + \mathbf{n}(t), \tag{71}$$

where $\mathbf{s}(t), \mathbf{n}(t)$ are independent and $E[\mathbf{n}(t)] = 0$, the PSD of output $\mathbf{y}(t)$ is given by

$$\begin{aligned} S_{\mathbf{y}}(f) &= S_{\mathbf{x}}(f) |H(f)|^2 \\ &= S_{\mathbf{s}}(f) |H(f)|^2 + S_{\mathbf{n}}(f) |H(f)|^2, \end{aligned} \tag{72}$$

where $S_{\mathbf{s}}(f) |H(f)|^2$ is signal output PSD and $S_{\mathbf{n}}(f) |H(f)|^2$ is noise PSD.

The input and output SNRs is given by

$$\text{SNR}_x = \frac{E[\mathbf{s}^2(t)]}{E[\mathbf{n}^2(t)]} = \frac{R_{\mathbf{s}}(0)}{R_{\mathbf{n}}(0)} = \frac{\int_{-\infty}^{\infty} S_{\mathbf{s}}(f)df}{\int_{-\infty}^{\infty} S_{\mathbf{n}}(f)df} \tag{73a}$$

$$\text{SNR}_y = \frac{\int_{-\infty}^{\infty} S_{\mathbf{s}}(f |H(f)|^2 df}{\int_{-\infty}^{\infty} S_{\mathbf{n}}(f) |H(f)|^2 df}. \tag{73b}$$

8.2 Gaussian Process

A Gaussian process $\mathbf{x}(t)$ a random process that for $\forall k > 0$ and for all times $t_1, \ldots, t_k$, the set of random variable $\mathbf{x}(t_1), \ldots, \mathbf{x}(t_k)$ is jointly Gaussian (i.e. described by Eq. (27)).

Properties:

- WSS Gaussian process is SSS.
- Gaussian process $\mathbf{x}(t)$ that passes through LTI system, $\mathbf{y}(t) = h(t) * \mathbf{x}(t)$, is also Gaussian process that may be described by the change of expectation and auto-correlation,

$$\begin{aligned} E[\mathbf{y}(t)] &= E[\mathbf{x}(t)] \int_{-\infty}^{\infty} h(s)ds \quad (74a) \\ &= E[\mathbf{x}(t)]H(0), \quad H(f) = \mathscr{F}\{h(t)\} \end{aligned}$$

$$C_{\mathbf{y}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \tag{74b}$$

- The resulting autocorrelation may be used for producing the correspondent covariance matrix $C_{\mathbf{Y}}$ of a multivariate Gaussian $\mathbf{Y} = [\mathbf{y}(t_1), \ldots, \mathbf{y}(t_N)]^T$

9 Poisson Process

- The Poisson process, $N(t)$, is described by

$$p\big(N(t) = k\big) = \exp(-\lambda t)\frac{(\lambda t)^k}{k!}, \; k = 0, 1, \ldots \tag{75a}$$
$$p\big(N(0) = 0\big) = 0 \tag{75b}$$
$$E[N(t)] = \lambda t \tag{75c}$$
$$Var[N(t)] = \lambda t \tag{75d}$$
$$p\big(N(t) \leqslant k\big) = \sum_{i=0}^{k} p\big(N(t) = k\big) \tag{75e}$$

- Independent & stationary increments:

  For any $t_4 > t_3 \geqslant t_2 > t_1$ and random variables I_1, I_2 defined by

$$I_1 = N(t_2) - N(t_1) \tag{76a}$$
$$I_2 = N(t_4) - N(t_3) \tag{76b}$$

  (a) I_1 and I_2 are *independent*
  (b) $t_2 - t_1 = t_4 - t_3 \Rightarrow I_1, I_2$ has the same distribution (*stationary* property)

- Time increment property

$$p\big(N(t_2) - N(t_1) = k\big) = p\big(N(t_2 - t_1) = k\big) \tag{77}$$

- Joint PMF ($t_2 > t_1$)

$$p\big(N(t_1) = i, N(t_2) = j\big) = \\ = p\big(N(t_1) = i\big) \cdot p\big(N(t_2 - t_1) = j - i\big) \quad (78)$$

- Conditional probability

$$p(N(t_1) = i | N(t_2) = j) = \\ = \frac{p\big(N(t_1) = i, N(t_2) = j\big)}{p\big(N(t_2) = j\big)} \quad (79)$$

- Special properties:

  * Given sum of two independent distributions $X \sim \mathcal{P}(\lambda_1)$ and $Y \sim \mathcal{P}(\lambda_2)$, the resulting distribution is given by $X + Y \sim \mathcal{P}(\lambda_1 + \lambda_2)$.

  – Sub-group of Poisson process is Poisson process.

  – Sum of two Poisson processes λ_1 and λ_2 is Poisson process $\lambda_1 + \lambda_2$ (but not a subtraction).

- Erlang: If $X_i \sim Exp(\lambda)$ is time difference between events, then

$$T_k = \sum_{i=1}^{k} X_i \sim Erlang(k, \lambda) \quad (80a)$$

$$E[T_k] = \frac{k}{\lambda} \quad (80b)$$

$$\mathrm{Var}[T_k] = \frac{k}{\lambda^2} \quad (80c)$$

9.1 Campbell Theorem

Given

$$z(t) = \sum_{k=1}^{\infty} \delta\left(t - T_k\right) \quad (81)$$

and casual system impulse response, $h(t)$, the resulting process is given by

$$y(t) = z(t) * h(t) = \sum_{k=1}^{\infty} h(t - T_k) \quad (82)$$

and the resulting statistics is given by

$$E[y(t)] = \lambda \int_0^t h(s) ds \quad (83a)$$

$$\mathrm{Var}[y(t)] = \lambda \int_0^t h^2(s) ds \quad (83b)$$

10 Different Supplementary Formulas

10.1 Derivatives

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\exp\left[f(x)\right] = \exp\left[f(x)\right]\frac{d}{dx}f(x)$$

10.2 Integrals

10.2.1 Indefinite

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1$$

$$\int \exp(ax)dx = \frac{1}{a}\exp(ax)$$

$$\int x\exp(ax)dx = \exp(ax)\left[\frac{x}{a} - \frac{1}{a^2}\right]$$

$$\int x^2\exp(ax)dx = \exp(ax)\left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right]$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

10.2.2 Definite

$$\int_0^\infty \exp(-a^2x^2)dx = \frac{\sqrt{\pi}}{2a}$$

$$\int_0^\infty x^2\exp(-a^2x^2)dx = \frac{\sqrt{\pi}}{4a^3}$$

$$\int_{-\infty}^\infty \delta(x)dx = 1$$

$$\int_{-\infty}^\infty f(x)\delta(x-a)dx = f(a)$$

10.3 Fourier Transform

10.3.1 Properties

$$\frac{d^n}{dt^n}f(t) \overset{\mathscr{F}}{\Longleftrightarrow} (j2\pi f)^n F(f)$$

$$f(-t) \overset{\mathscr{F}}{\Longleftrightarrow} F^*(f)$$

$$f(t-t_0) \overset{\mathscr{F}}{\Longleftrightarrow} F(f)e^{-j2\pi f t_0}$$

$$f(t)e^{j2\pi f_0 t} \overset{\mathscr{F}}{\Longleftrightarrow} F(f-f_0)$$

10.3.2 Transform pairs

$$u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2}\left(\frac{1}{j\pi f} + \delta(f)\right)$$

$$\exp(-at)u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{a+j2\pi f}$$

$$t\exp(-at)u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{(a+j2\pi f)^2}$$

$$\exp(-a|t|) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{2a}{a^2+4\pi^2f^2}$$

$$\exp(-at^2) \overset{\mathscr{F}}{\Longleftrightarrow} \sqrt{\frac{\pi}{a}}\exp\left(-\frac{(\pi f)^2}{a}\right)$$

$$\cos(2\pi f_a t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2}\left[\delta(f-f_a) + \delta(f+f_a)\right]$$

$$\sin(2\pi f_a t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2j}\left[\delta(f-f_a) - \delta(f+f_a)\right]$$

10.4 Trigonometry

$$\sin^2(\alpha) = \frac{1}{2}\left(1-\cos(2\alpha)\right)$$

$$\cos^2(\alpha) = \frac{1}{2}\left(1+\cos(2\alpha)\right)$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin\left((\alpha+\beta)\right)\right]$$

10.5 Matrices

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$$\det[\mathbf{A}] = ad - bc$$

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

10.5.1 Eigenvalues/vectors

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\det[\mathbf{A} - \lambda\mathbf{I}] = 0 \Rightarrow \mathbf{\Lambda}$$

$$\mathbf{AV} = \mathbf{\Lambda V}$$