המחלקה להנדסת חשמל ואלקטרוניקה

תאריך הבחינה: 26.01.18

שעות הבחינה: 8:30-11:30

מבוא לאותות אקראיים

מועד א'

ד"ר דימה בחובסקי, אביתר רימון

תשע"ח סמסטר א'

חומר עזר - עד 2 דפי נוסחאות אישיים (משני צדדים), מחשבון

הוראות מיוחדות:

- יש לציין באופן מלא וברור את שלבי הפתרון. תשובה ללא הסבר לא תתקבלנה.
- אם לא מצויין אחרת, הסעיפים הם בעלי ניקוד זהה.

השאלון כולל 10 דפים (כולל דף זה)

בהצלחה!

1 משתנים רב-ממדים (30 נק')

נתונים משתנים אקראיים $X_1 \sim N(m,\sigma^2), X_2 \sim N(m,\sigma^2)$. **לא ניתן להניח אי תלות בין** X_1, X_2**!** נגדיר משתנים חדשים:

$$V = X_1 \cos(\theta) + X_2 \sin(\theta) \tag{1}$$

$$W = -X_1 \sin(\theta) + X_2 \cos(\theta) \tag{2}$$

(א) מצא ערך של $-\pi \leqslant \theta \leqslant \pi$, עבורו משתנים V, W בלתי תלויים.

(ב) (סעיף **כפול**!) מה הפילוג המשותף של של משתנים V, W, ומה הפרמטרים של הפלוג הזה? בסעיף זה ניתן להניח $\mathrm{Cov}[X_1, X_2] = b$.

2 סטציואנריות (מתוך Kay דוגמה בעמוד 656) (30 נק')

נתונים זוג תהליכים אקראיים $\mathbf{x}[n] = A, \mathbf{y}[n] = (-1)^n A$, כאשר A הינו משתנה אקראי בעל תוחלת 0 ושונות 1.

(א) האם תהליך $\mathbf{x}[n]$ הינו WSS? **הוכח/נמק!**

(ב) האם תהליך $\mathbf{y}[n]$ הינו WSS? **הוכח/נמק!**

(ג) האם תהליכים $\mathbf{x}[n], \mathbf{y}[n]$ הינם WSS במשותף? **הוכח/נמק!**

3 חיזוי לינארי (מתבסס על Kay 18.22, עמוד 634) (20 נק')

נתון תהליך אקראי $\mathbf{x}[n] = a\mathbf{x}[n-1] + \mathbf{w}[n]$, כאשר $\mathbf{w}[n]$ הינו רעש לבן גאוסי בעל $\sigma_w^2 = 1$.

ידוע (ע"פ Kay Example 17.5), שפונקציית אוטו-קרלציה של התהליך נתונה ע"י

$$R_{\mathbf{x}}[k] = \frac{1}{1-a^2} a^{|k|} \tag{3}$$

נדרש לעשות חיזוי לינארי עבור התהליך מהצורה $\hat{\mathbf{x}}[n+1] = b_1\mathbf{x}[n] + b_2\mathbf{x}[n-1]$.

מצא ערכים אופטימליים של b_1, b_2 במובן שגיאה ריבועית מינימלית (MMSE), ונמק את התוצאה המתקבלת. חשב את שגיאת חיזוי המתקבלת.

4 Poisson (מתוך Kay, 21.13) (20 נק')

קצב הגעת מוניות ממוצע הוא 1 לדקה.

(א) מהו הסיכוי ליותר מ-2 מוניות בדקה מסויימת?

(ב) מהו הסיכוי לחכות למונית **פחות מדקה**, אחרי המתנה למונית במשך 10 דק'?

Random Processes – Formulas

1 Random Variables

1.1 Distributions

	Notation	PDF/PMF	CDF	$E[X]$	$\mathrm{Var}[X]$
Bernoulli	Ber(p)	$\begin{cases} 1-p & k=0 \\ p & k=1 \end{cases}$	$\begin{cases} 0 & x<0 \\ 1-p & 0\leqslant x<1 \\ 1 & 1\leqslant x \end{cases}$	p	$p(1-p)$
Uniform	U[a,b]	$\frac{1}{b-a}, a\leqslant x\leqslant b$	$\begin{cases} 0 & x<a \\ \frac{x-a}{b-a} & a\leqslant x\leqslant b \\ 1 & b<x \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal	$N(\mu,\sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$	$\Phi(x)$	μ	σ^2
Exponential	$Exp(\lambda)$	$\lambda\exp(-\lambda x), x\geqslant 0$	$1-\exp(-\lambda x)$	$1/\lambda$	$1/\lambda^2$
Poisson	$\mathcal{P}(\lambda)$	$p(X=k)=\exp(-\lambda)\frac{\lambda^k}{k!}$	$\exp(-\lambda)\sum_{i=0}^{k}\frac{\lambda^i}{i!}$	λ	λ
Erlang	$Erlang(k,\lambda t)$	$\lambda\frac{(\lambda t)^{k-1}}{(k-1)!}\exp(-\lambda t)$	$1-\exp(-\lambda t)\sum_{n=0}^{k-1}\frac{(\lambda t)^n}{n!}$	$\frac{k}{\lambda t}$	$\frac{k}{(\lambda t)^2}$

1.2 Properties

Definitions:

$$F_X(x) = p(X\leqslant x) \tag{1a}$$
$$f_X(x) = \frac{\partial F_X(x)}{\partial x} \tag{1b}$$
$$F_X(x) = \int_{-\infty}^{x} f_X(p)\,dp \tag{1c}$$
$$p(a<X\leqslant b) = F_X(b)-F_X(a) \tag{1d}$$

$$p_X[x_k] = p(X=x_k) \tag{2a}$$
$$F_X(x) = \sum_{k:x_k\leqslant x} p_X[x_k] \tag{2b}$$

Expectation:

$$E[X] = \begin{cases} \int_{-\infty}^{\infty} x f_X(x)dx \\ \sum_i x_i p_x[x_i] \end{cases} \tag{3a}$$
$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x)dx \\ \sum_i g(x_i) p_x[x_i] \end{cases} \tag{3b}$$
$$E[aX+b] = aE[x]+b \tag{3c}$$

Variance:

$$\begin{aligned}\mathrm{Var}[X] &= E[(X-E[X])^2] \\ &= E[X^2]-E^2[X]\end{aligned} \tag{4a}$$
$$\mathrm{Var}[aX+b] = a^2\,\mathrm{Var}[X] \tag{4b}$$

2 Two Random Variables

2.1 Joint Distributions

Definitions:

$$F_{XY}(x,y) = p(X\leqslant x, Y\leqslant y) \tag{5a}$$
$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x\partial y} \geqslant 0 \tag{5b}$$
$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(s,p)\,dp\,ds \tag{5c}$$

$$p[x_j,y_k] = p(X=x_j, Y=y_k) \tag{6a}$$
$$F_{XY}(x,y) = p(X\leqslant x_j, Y\leqslant y_k) \tag{6b}$$

Conditional distribution (Bayes)
$(f_X(x), f_Y(y), p_X[x_k], p_Y[y_k] > 0)$:

$$f_{Y|X}(y|x)f_X(x) = f_{X|Y}(x|y)f_Y(x) = f_{XY}(x,y) \quad (7a)$$

$$p_{Y|X}[y_j|x_k]p_X[x_k] = p_{X|Y}[x_k|y_j]p_Y[y_j] = p_{XY}[x_k,y_j] \quad (7b)$$

$$F_{Y|X}(y|x) = p(Y \leqslant y|X = x) = \int_{-\infty}^{y} f_{Y|X}(s|x)ds \quad (7c)$$

$$F_{Y|X}[y|x_k] = \frac{p[Y \leqslant y_j, X = x_k]}{p_X[x_k]} \quad (7d)$$

Expectation:

$$E[XY] = \iint xyf_{XY}(x,y)dxdy \quad (8a)$$

$$E[g(X)] = \begin{cases} \iint g(x,y)f_{XY}(x,y)dxdy \\ \sum_i \sum_k g(x_i,y_k)p_x[x_i,y_k] \end{cases} \quad (8b)$$

$$E[aX + bY] = aE[X] + bE[Y] \quad (8c)$$

Conditional expectation & Variance:

$$E[Y|X] = \begin{cases} \int yf_{Y|X}(y|x)dy \\ \sum_j y_j p[y_j|x_k] \end{cases} \quad (9a)$$

$$E[X] = E[E[X|Y]] = \iint yf_{Y|X}(y|x)f_X(x)dxdy \quad (9b)$$

$$\mathrm{Var}[Y|X] = E[Y^2|X] - E^2[Y|X] \quad (9c)$$

$$\mathrm{Var}[Y] = \mathrm{Var}[E[Y|X]] + E[\mathrm{Var}[Y|X]] \quad (9d)$$

Independent random variables:

$$f_{XY}(x,y) = f_X(x)f_Y(y) \quad (10a)$$

$$p_{XY}[x_k,y_j] = p_X[x_k]p_Y[y_j] \quad (10b)$$

$$F_{XY}(x,y) = F_X(x)F_Y(y) \quad (10c)$$

$$E[XY] = E[X]E[Y] \quad (10d)$$

$$E[g_1(X)g_2(Y)] = E[g_1(X)]E[g_2(Y)] \quad (10e)$$

$$\mathrm{Var}[aX + bY] = a^2\,\mathrm{Var}[X] + b^2\,\mathrm{Var}[Y] \quad (10f)$$

Marginal distribution:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy \quad (11a)$$

$$p_X[x_k] = \sum_j p_{XY}[x_k,y_j] \quad (11b)$$

$$F_X(x) = F_{XY}(x,\infty) \quad (11c)$$

$$F_Y(y) = F_{XY}(\infty,y) \quad (11d)$$

2.2 Correlation, Covariance & Correlation Coefficient

- For two jointly-distributed random variables X and Y, covariance is given by

$$\begin{aligned}\mathrm{Cov}[X,Y] =& E[(X - E[X])(Y - E[Y])] \\ =& E[XY] - E[X]E[Y].\end{aligned} \quad (12)$$

Main covariance properties are:

$$\mathrm{Cov}[X,X] = Var[X] \quad (13a)$$

$$\mathrm{Cov}[X,Y] = \mathrm{Cov}[Y,X] \quad (13b)$$

$$\mathrm{Cov}[X,a] = 0 \quad (13c)$$

$$\mathrm{Cov}[aX,bY] = ab\,\mathrm{Cov}[X,Y] \quad (13d)$$

$$\mathrm{Cov}[X,Y] = \mathrm{Cov}[X+a,Y+b] \quad (13e)$$

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X,Y] \quad (13f)$$

- Correlation coefficient (also termed as Pearson product-moment correlation coefficient) is given by

$$\rho_{XY} = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} \quad (14)$$

such that $|\rho_{XY}| \leqslant 1$.

2.3 Relations

- When X and Y are *orthogonal*, $E[XY] = 0$.
- When X and Y are *uncorrelated*, $\mathrm{Cov}[X,Y] = \rho_{XY} = 0$.
- When X and Y are *independent*, they are also uncorrelated (see also Eqs. 10).
- When X and Y are *jointly* Gaussian and uncorrelated $\Rightarrow$ X and Y are independent.
- Joint$\Rightarrow$ marginal, marginal $\not\Rightarrow$ joint

3 Random Processes – General Properties

- PDF & CDF

$$F_{\mathbf{x}}(x;t) = p(\mathbf{x}(t) \leqslant x) \quad (15a)$$

$$f_{\mathbf{x}}(x;t) = \frac{\partial}{\partial x}F_{\mathbf{x}}(x;t) \quad (15b)$$

- Average:

$$E[\mathbf{x}(t)] = \int_{-\infty}^{\infty} xf_{\mathbf{x}}(x;t)dx \quad (16a)$$

$$E[\mathbf{x}[n]] = \int_{-\infty}^{\infty} xf_{\mathbf{x}}(x;n)dx \quad (16b)$$

- Variance:

$$Var[\mathbf{x}(t)] = E[\mathbf{x}^2(t)] - E^2[\mathbf{x}(t)] \tag{17a}$$
$$Var[\mathbf{x}[n]] = E[\mathbf{x}^2[n]] - E^2[\mathbf{x}[n]] \tag{17b}$$

- Auto-correlation

$$R_{\mathbf{x}}(t_1, t_2) = E[\mathbf{x}(t_1)\mathbf{x}(t_2)] \tag{18a}$$
$$R_{\mathbf{x}}(t, t+\tau) = E[\mathbf{x}(t)\mathbf{x}(t+\tau)] \tag{18b}$$
$$R_{\mathbf{x}}(t_1, t_2) = R_{\mathbf{x}}(t_2, t_1) \tag{18c}$$
$$R_{\mathbf{x}}(t, t) = E[\mathbf{x}^2(t)] \tag{18d}$$
$$R_{\mathbf{x}}[n_1, n_2] = E[\mathbf{x}[n_1]\mathbf{x}[n_2]] \tag{18e}$$

- Auto-covariance

$$\begin{aligned} C_{\mathbf{x}}(t_1, t_2) &= R_{\mathbf{x}}(t_1, t_2) - E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)] \\ &= E[\{\mathbf{x}(t_1) - E[\mathbf{x}(t_1)]\}\{\mathbf{x}(t_2) - E[\mathbf{x}(t_2)]\}] \end{aligned} \tag{19}$$

$$C_{\mathbf{x}}(t, t) = Var[\mathbf{x}(t)] \tag{20}$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(t_1, t_2) = \frac{C_{\mathbf{x}}(t_1, t_2)}{\sqrt{C_{\mathbf{x}}(t_1, t_1)C_{\mathbf{x}}(t_2, t_2)}} \tag{21a}$$
$$|\rho_{\mathbf{x}}(t_1, t_2)| \leqslant 1 \tag{21b}$$

- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *orthogonal*, $R_{\mathbf{x}}(t_1, t_2) = 0$.
- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *uncorrelated*, $C_{\mathbf{x}}(t_1, t_2) = \rho_{\mathbf{x}}(t_1, t_2) = 0$.
- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *independent*, $R_{\mathbf{x}}(t_1, t_2) = E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)]$.

4 Wide-Sense Stationary (WSS) Process

Definition:

$$E[\mathbf{x}(t)] = E[\mathbf{x}(0)] = \mu_{\mathbf{x}} = \text{const} \tag{22a}$$
$$R_{\mathbf{x}}(t_1, t_2) = R_{\mathbf{x}}(\tau = |t_2 - t_1|), \quad \forall t_1, t_2 \tag{22b}$$
$$E[\mathbf{x}[n]] = E[\mathbf{x}[0]] = \mu_{\mathbf{x}} = \text{const} \tag{22c}$$
$$R_{\mathbf{x}}[n_1, n_2] = R_{\mathbf{x}}(k = |n_2 - n_1|), \quad \forall n_1, n_2 \tag{22d}$$

- Auto-correlation

$$R_{\mathbf{x}}(\tau) = E[\mathbf{x}(t)\mathbf{x}(t+\tau)] \tag{23a}$$
$$R_{\mathbf{x}}[k] = E[\mathbf{x}[n]\mathbf{x}(n+k)] \tag{23b}$$

Properties:

$$R_{\mathbf{x}}(-\tau) = R_{\mathbf{x}}(\tau) \tag{24a}$$
$$R_{\mathbf{x}}(0) = E[|\mathbf{x}(0)|^2] = E[|\mathbf{x}(t)|^2] \tag{24b}$$
$$Var[\mathbf{x}(t)] = C_{\mathbf{x}}(0) = \sigma_{\mathbf{x}}^2 \tag{24c}$$
$$|R_{\mathbf{x}}(0)| \geqslant R_{\mathbf{x}}(\tau) \tag{24d}$$

Deterministic definition ($x[n]$ is <u>not</u> random)

$$R_x(\tau) = x(\tau) * x(-\tau) \tag{25a}$$
$$R_x[k] = x[k] * x[-k] \tag{25b}$$

- Auto-covariance

$$C_{\mathbf{x}}(\tau) = R_{\mathbf{x}}(\tau) - \mu_{\mathbf{x}}^2 \tag{26a}$$
$$C_{\mathbf{x}}[k] = R_{\mathbf{x}}[k] - \mu_{\mathbf{x}}^2 \tag{26b}$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(\tau) = \frac{C_{\mathbf{x}}(\tau)}{C_{\mathbf{x}}(0)} \tag{27a}$$
$$\rho_{\mathbf{x}}[k] = \frac{C_{\mathbf{x}}[k]}{C_{\mathbf{x}}[0]} \tag{27b}$$

- Power spectral density (PSD)

$$\begin{aligned} S_{\mathbf{x}}(f) &= \mathcal{F}\{R_{\mathbf{x}}(\tau)\} = \\ &= \int_{-\infty}^{\infty} R_{\mathbf{x}}(\tau) \exp(j2\pi f\tau)\, d\tau && (28a) \\ &= 2\int_{0}^{\infty} R_{\mathbf{x}}(\tau) \cos(2\pi f\tau)\, d\tau && (28b) \\ P_{\mathbf{x}}(f) &= \mathcal{F}\{R_{\mathbf{x}}[k]\} = \qquad -1/2 \leqslant f \leqslant 1/2 \\ &= \sum_{k=-\infty}^{\infty} R_{\mathbf{x}}[k] \exp(-j2\pi fk) && (28c) \\ &= 2\sum_{k=0}^{\infty} R_{\mathbf{x}}[k] \cos(-2\pi fk) && (28d) \end{aligned}$$

Properties:

$$S_{\mathbf{x}}(f) = S_{\mathbf{x}}(-f) \tag{29a}$$
$$S_{\mathbf{x}}(f) \geqslant 0,\ \forall f \tag{29b}$$
$$S_{\mathbf{x}}(f) \in \mathbb{R} \quad \text{(real numbers)} \tag{29c}$$
$$P_{\mathbf{x}}(f) = P_{\mathbf{x}}(-f) \tag{29d}$$
$$P_{\mathbf{x}}(f) \geqslant 0,\ \forall f \tag{29e}$$
$$P_{\mathbf{x}}(f) \in \mathbb{R} \quad \text{(real numbers)} \tag{29f}$$
$$P_{\mathbf{x}}(f) = P_{\mathbf{x}}(f+1) \tag{29g}$$

Average power

$$P_{\mathbf{x}} = E[\mathbf{x}^2(t)] = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(f) df \tag{30a}$$
$$= E[\mathbf{x}^2[n]] = R_{\mathbf{x}}[0] = \int_{-1/2}^{1/2} P_{\mathbf{x}}(f) df \tag{30b}$$

Deterministic definition:

$$S_x(f) = X(f)X^*(f) = |X(f)|^2 \tag{31}$$

4.1 White Noise & White Gaussian Noise (WGN) Process

White noise process is SSS (WSS) process that is characterized by

$$R_{\mathbf{n}}(\tau) = \sigma^2\delta(\tau) \tag{32a}$$
$$R_{\mathbf{n}}[k] = \sigma^2\delta[k] \tag{32b}$$
$$S_{\mathbf{n}}(f) = \sigma^2 \quad \forall f \tag{32c}$$

For WGN process, $\mathbf{n}(t) \sim N(0, \sigma^2)$,

$$R_{\mathbf{n}}(\tau) = \frac{N_0}{2}\delta(\tau) \tag{33a}$$
$$S_{\mathbf{n}}(f) = \frac{N_0}{2} \quad \forall f \tag{33b}$$

5 Cross-Signal

- Cross-correlation

$$R_{\mathbf{xy}}(t_1, t_2) = E[\mathbf{x}(t_1)\mathbf{y}(t_2)] \tag{34}$$

- Cross-covariance

$$C_{\mathbf{xy}}(t_1, t_2) = R_{\mathbf{xy}}(t_1, t_2) - E[\mathbf{x}(t_1)]E[\mathbf{y}(t_2)] \tag{35}$$

- Correlation Coefficient

$$\rho_{\mathbf{xy}}(t_1, t_2) = \frac{C_{\mathbf{xy}}(t_1, t_2)}{\sqrt{C_{\mathbf{xy}}(t_1, t_1)C_{\mathbf{xy}}(t_2, t_2)}} \tag{36}$$

5.1 WSS Cross-signal

- $\mathbf{x}(t), \mathbf{y}(t)$ are jointly WSS, if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ each of them is WSS and

$$R_{\mathbf{xy}}(\tau) = E[\mathbf{x}(t)\mathbf{y}(t+\tau)] \tag{37}$$

- When $\mathbf{x}(t)$ and $\mathbf{y}(t+\tau)$ are *uncorrelated jointly WSS*, $C_{\mathbf{xy}}(\tau) = 0$.

  Properties

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{yx}}(-\tau) \tag{38a}$$
$$|R_{\mathbf{xy}}(\tau)| \leqslant \sqrt{R_{\mathbf{x}}(0)R_{\mathbf{y}}(0)} \tag{38b}$$
$$|R_{\mathbf{xy}}(\tau)| \leqslant \frac{1}{2}\left[R_{\mathbf{x}}(0) + R_{\mathbf{y}}(0)\right] \tag{38c}$$

  Deterministic definition

$$R_{xy}(\tau) = x(\tau) * y(-\tau) \tag{39}$$

- Cross-covariance

$$C_{\mathbf{xy}}(\tau) = R_{\mathbf{xy}}(\tau) - \mu_{\mathbf{x}}\mu_{\mathbf{y}} \tag{40}$$

- Cross-PSD

$$S_{\mathbf{xy}}(f) = \mathcal{F}\{R_{\mathbf{xy}}(\tau)\} \tag{41}$$

  Properties

$$S_{\mathbf{xy}}(f) = S_{\mathbf{yx}}(-f) = S^*_{\mathbf{xy}}(-f) \tag{42}$$

  Deterministic definition

$$S_{xy}(f) = X(f)Y^*(f) \tag{43}$$

- Coherence

$$\gamma_{\mathbf{xy}}(f) = \frac{S_{\mathbf{xy}}(f)}{\sqrt{S_{\mathbf{x}}(f)S_{\mathbf{y}}(f)}} \tag{44}$$

6 LTI and WSS Random Process

Output of LTI system with impulse response $h(t)$ and random process $x(t)$,

$$y(t) = x(t) * h(t) \tag{45}$$

Average

$$m_{\mathbf{y}} = m_{\mathbf{x}}\int_{-\infty}^{\infty} h(s)ds = m_{\mathbf{x}}H(f=0) \tag{46}$$

Cross-correlation & cross-covariance:

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) \tag{47a}$$
$$C_{\mathbf{xy}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) \tag{47b}$$
$$R_{\mathbf{yx}}(\tau) = R_{\mathbf{x}}(\tau) * h(-\tau) \tag{47c}$$
$$C_{\mathbf{yx}}(\tau) = C_{\mathbf{x}}(\tau) * h(-\tau) \tag{47d}$$
$$R_{\mathbf{y}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \tag{47e}$$
$$C_{\mathbf{y}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \tag{47f}$$

Power-Spectral Density (PSD) & Cross-PSD: Given frequency response $H(f) = \mathcal{F}\{h(\tau)\},\ H^*(f) = \mathcal{F}\{h(-\tau)\}$

$$S_{\mathbf{xy}}(f) = S_{\mathbf{x}}(f)H(f) \tag{48a}$$
$$S_{\mathbf{yx}}(f) = S_{\mathbf{x}}(f)H^*(f) \tag{48b}$$
$$S_{\mathbf{y}}(f) = S_{\mathbf{x}}(f)H(f)H^*(f) = S_{\mathbf{x}}(f)|H(f)|^2 \tag{48c}$$

Power of the process:

$$P_x = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(f)\,df \tag{49a}$$
$$P_y = R_{\mathbf{y}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(f)|H(f)|^2 df \tag{49b}$$

6.1 SNR

Given input signal

$$\mathbf{x}(t) = \mathbf{s}(t) + \mathbf{n}(t), \tag{50}$$

where $\mathbf{s}(t), \mathbf{n}(t)$ are independent and $E[\mathbf{n}(t)] = 0$, the PSD of output $\mathbf{y}(t)$ is given by

$$\begin{aligned} S_{\mathbf{y}}(f) &= S_{\mathbf{x}}(f)\left|H(f)\right|^2 \\ &= S_{\mathbf{s}}(f)\left|H(f)\right|^2 + S_{\mathbf{n}}(f)\left|H(f)\right|^2, \end{aligned} \tag{51}$$

where $S_{\mathbf{s}}(f)\left|H(f)\right|^2$ is signal output PSD and $S_{\mathbf{n}}(f)\left|H(f)\right|^2$ is noise PSD.

The input and output SNRs is given by

$$\mathrm{SNR}_x = \frac{E[\mathbf{s}^2(t)]}{E[\mathbf{n}^2(t)]} = \frac{R_{\mathbf{s}}(0)}{R_{\mathbf{n}}(0)} = \frac{\int_{-\infty}^{\infty} S_{\mathbf{s}}(f)df}{\int_{-\infty}^{\infty} S_{\mathbf{n}}(f)df} \tag{52a}$$

$$\mathrm{SNR}_y = \frac{\int_{-\infty}^{\infty} S_{\mathbf{s}}(f\left|H(f)\right|^2 df}{\int_{-\infty}^{\infty} S_{\mathbf{n}}(f)\left|H(f)\right|^2 df}. \tag{52b}$$

7 Multi-dimensional processes

7.1 Covariance matrix

Given random vector $\mathbf{X} = (X_1, X_2, \ldots, X_N)^T$,

$$\begin{aligned} \mathrm{C}_{\mathbf{X}} =& \mathrm{Cov}[\mathbf{X}, \mathbf{X}] = E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])^T] \\ =& E[\mathbf{X}\mathbf{X}^T] - E[\mathbf{X}]E[\mathbf{X}]^T \\ =& \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1, X_2] & \cdots & \mathrm{Cov}[X_1, X_N] \\ \mathrm{Cov}[X_2, X_1] & \mathrm{Var}[X_2] & \cdots & \mathrm{Cov}[X_2, X_N] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}[X_N, X_1] & \mathrm{Cov}[X_N, X_2] & \cdots & \mathrm{Var}[X_N] \end{bmatrix} \end{aligned} \tag{53}$$

Properties:

- Symmetry

$$\mathrm{C}_{\mathbf{X}} = \mathrm{C}_{\mathbf{X}}^T \quad \mathrm{Cov}[X_i, X_j] = \mathrm{Cov}[X_j, X_i] \tag{54}$$

- Variance of linear combination: Given vector $\mathbf{a} = (a_1, a_2, \ldots, a_N)^T$,

$$\mathrm{Var}[\mathbf{a}^T\mathbf{X}] = \mathbf{a}^T C_{\mathbf{X}}\mathbf{a} \tag{55}$$

- Linear transformation: Given linear tranformation $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$,

$$E[\mathbf{Y}] = \mathbf{A}E[\mathbf{X}] + \mathbf{b} \tag{56a}$$

$$C_{\mathbf{Y}} = \mathbf{A}C_{\mathbf{X}}\mathbf{A}^T \tag{56b}$$

- Uncorrelated variables

$$\mathrm{C}_{\mathbf{X}} = \mathrm{diag}\left[\mathrm{Var}[X_1], \mathrm{Var}[X_2], \ldots, \mathrm{Var}[X_N]\right] \tag{57}$$

- Cross-covariance: For two random vectors $\mathbf{X} \in \mathbb{R}^m$ and $\mathbf{Y} \in \mathbb{R}^n$, the resulting $m \times n$ cross-covariance matrix is given by

$$\begin{aligned} \mathrm{Cov}[\mathbf{X}, \mathbf{Y}] =& C_{\mathbf{X}\mathbf{Y}} \\ =& E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{Y} - E[\mathbf{Y}])^T] \\ =& E[\mathbf{X}\mathbf{Y}^T] - E[\mathbf{X}]E[\mathbf{Y}]^T \end{aligned} \tag{58}$$

$$\mathrm{C}_{\mathbf{Y}\mathbf{X}} = \mathrm{C}_{\mathbf{X}\mathbf{Y}}^T \tag{59}$$

7.2 Relation Between Covariance Matrix & Auto-covariance

Given WSS process $\mathbf{x}(t)$, the corresponding correlation matrix of $\mathbf{X} = [\mathbf{x}(t_1), \ldots, \mathbf{x}(t_N)]^T$ is given by

$$R_{\mathbf{X}} = E\left[\mathbf{X}\mathbf{X}^T\right] \tag{60}$$

$$R_{\mathbf{X}}(i, j) = E\left[X_i X_j\right] = R_{\mathbf{x}}\left(|t_i - t_j|\right) \tag{61}$$

7.3 MMSE Linear Prediction

Mean square error (MSE) of predictor $\hat{Y}$ is given by

$$mse = E[(Y - \hat{Y})^2] \tag{62}$$

- Given two random variables, X, Y, and predictor

$$\hat{Y} = aX + b, \tag{63}$$

minimum MSE (MMSE) predictor given $X = x$ is

$$\hat{Y} = E[Y] + \frac{\mathrm{Cov}[X, Y]}{\mathrm{Var}[X]}\left(x - E[X]\right) \tag{64}$$

- Given N samples of process $\mathbf{x}[n]$, and predictor

$$\hat{\mathbf{x}}[n+1] = \sum_{i=1}^{N} a_i \mathbf{x}[n-i+1], \tag{65}$$

the values of a_i are given by solution of

$$\begin{bmatrix} R_{\mathbf{x}}[0] & R_{\mathbf{x}}[1] & \cdots & R_{\mathbf{x}}[N-1] \\ R_{\mathbf{x}}[1] & R_{\mathbf{x}}[0] & \cdots & R_{\mathbf{x}}[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ R_{\mathbf{x}}[N-1] & R_{\mathbf{x}}[N-2] & \cdots & R_{\mathbf{x}}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} R_{\mathbf{x}}[1] \\ R_{\mathbf{x}}[2] \\ \vdots \\ R_{\mathbf{x}}[N] \end{bmatrix} \tag{66}$$

and the resulting MMSE is

$$mmse = R_{\mathbf{x}}[0] - \sum_{i=1}^{N} a_i R_{\mathbf{x}}[i] \tag{67}$$

8 Gaussian Variables & Processes

8.1 Bi-variate & Multivariate Normal Distribution

Joint Gaussian distribution of X_1 and X_2

$$f_{X_1 X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right]\right) \tag{68}$$

Multivariate Gaussian distribution of $\mathbf{X} = (X_1, X_2, \ldots, X_N)^T$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{N/2} \det\left[\mathrm{C}_{\mathbf{X}}\right]} \exp\left\{-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu}\right)^T \mathrm{C}_{\mathbf{X}}^{-1} \left(\mathbf{x}-\boldsymbol{\mu}\right)\right\}, \tag{69}$$

Properties:

- Linear combination of Gaussian variables is Gaussian variable,
- Linear transformation – follows Eqs. (56a).
- If jointly distributed Gaussian random variables are *uncorrelated*, they are also *independent*

8.2 Gaussian Process

A Gaussian process $\mathbf{x}(t)$ a random process that for $\forall k > 0$ and for all times $t_1, \ldots, t_k$, the set of random variable $\mathbf{x}(t_1), \ldots, \mathbf{x}(t_k)$ is jointly Gaussian (i.e. described by Eq. (69)).

Properties:

- WSS Gaussian process is SSS.
- Gaussian process $\mathbf{x}(t)$ that passes through LTI system, $\mathbf{y}(t) = h(t) * \mathbf{x}(t)$, is also Gaussian process but with corresponding change in expectation and auto-covariance function.

9 Poisson Process

- The Poisson process, $N(t)$, is described by

$$p\big(N(t)=k\big) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \quad k = 0, 1, \ldots \tag{70a}$$

$$p\big(N(0)=0\big) = 0 \tag{70b}$$

$$E[N(t)] = \lambda t \tag{70c}$$

$$Var[N(t)] = \lambda t \tag{70d}$$

$$p\big(N(t) \leqslant k\big) = \sum_{i=0}^{k} p\big(N(t)=k\big) \tag{70e}$$

- Independent & stationary increments:
  For any $t_4 > t_3 \geqslant t_2 > t_1$ and random variables I_1, I_2 defined by

$$I_1 = N(t_2) - N(t_1) \tag{71a}$$

$$I_2 = N(t_4) - N(t_3) \tag{71b}$$

  (a) I_1 and I_2 are *independent*

(b) $t_2 - t_1 = t_4 - t_3 \Rightarrow I_1, I_2$ has the same distribution (*stationary* property)

- Time increment property

$$p\big(N(t_2) - N(t_1) = k\big) = p\big(N(t_2 - t_1) = k\big) \quad (72)$$

- Joint PMF ($t_2 > t_1$)

$$\begin{aligned} &p\big(N(t_1) = i, N(t_2) = j\big) = \\ &\quad = p\big(N(t_1) = i\big) \cdot p\big(N(t_2 - t_1) = j - i\big) \end{aligned} \quad (73)$$

- Conditional probability

$$\begin{aligned} &p(N(t_1) = i | N(t_2) = j) = \\ &\quad = \frac{p\big(N(t_1) = i, N(t_2) = j\big)}{p\big(N(t_2) = j\big)} \end{aligned} \quad (74)$$

- Special properties:

  * Given sum of two independent distributions $X \sim \mathcal{P}(\lambda_1)$ and $Y \sim \mathcal{P}(\lambda_2)$, the resulting distribution is given by $X + Y \sim \mathcal{P}(\lambda_1 + \lambda_2)$.

  – Sub-group of Poisson process is Poisson process.

  – Sum of two Poisson processes λ_1 and λ_2 is Poisson process $\lambda_1 + \lambda_2$ (but not a subtraction).

- Erlang: If $X_i \sim Exp(\lambda)$ is time difference between events, then

$$T_k = \sum_{i=1}^{k} X_i \sim Erlang(k, \lambda) \quad (75a)$$

$$E[T_k] = \frac{k}{\lambda} \quad (75b)$$

$$\mathrm{Var}[T_k] = \frac{k}{\lambda^2} \quad (75c)$$

9.1 Campbell Theorem

Given

$$z(t) = \sum_{k=1}^{\infty} \delta\left(t - T_k\right) \quad (76)$$

and casual system impulse response, $h(t)$, the resulting process is given by

$$y(t) = z(t) * h(t) = \sum_{k=1}^{\infty} h(t - T_k) \quad (77)$$

and the resulting statistics is given by

$$E[y(t)] = \lambda \int_0^t h(s)ds \quad (78a)$$

$$\mathrm{Var}[y(t)] = \lambda \int_0^t h^2(s)ds \quad (78b)$$

10 Markov Chain

- Transition matrix

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}, \quad (79)$$

where $p_{ij} = p\big(\mathbf{x}[n] = j | \mathbf{x}[n-1] = i\big)$

- Chapman-Kolmogorov equation

$$\mathbf{p}^T[n_1 + n] = \mathbf{p}^T[n_1]\mathbf{P}^n, \quad (80)$$

where $\mathbf{p}[n]$ is state probability vector

$$\mathbf{p}[n] = \begin{bmatrix} p_0[n] \\ p_1[n] \end{bmatrix} \quad (81)$$

and $p_i[n] = p\big(\mathbf{x}[n] = i\big)$, $i = 0, 1$.

- For general transition matrix of the form

$$\mathbf{P} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \quad (82)$$

$$\begin{aligned} \mathbf{P}^n = &\begin{bmatrix} \dfrac{\beta}{\alpha+\beta} & \dfrac{\alpha}{\alpha+\beta} \\ \dfrac{\beta}{\alpha+\beta} & \dfrac{\alpha}{\alpha+\beta} \end{bmatrix} + \\ &+ (1-\alpha-\beta)^n \begin{bmatrix} \dfrac{\alpha}{\alpha+\beta} & -\dfrac{\alpha}{\alpha+\beta} \\ -\dfrac{\beta}{\alpha+\beta} & \dfrac{\beta}{\alpha+\beta} \end{bmatrix} \end{aligned} \quad (83)$$

- Steady-state probability vector

$$\boldsymbol{\pi}^T = \begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix}, \quad (84)$$

where $\pi_i = \lim_{n\to\infty} p\big(\mathbf{x}[n] = i\big)$, $i = 0, 1$

10.1 Ergodic Markov chain

For $\mathbf{P}^n > 0$,

$$\boldsymbol{\pi}^T = \boldsymbol{\pi}^T\mathbf{P} \quad (85)$$

Average number of time-steps to return to state i from the last occurrence of state i is $1/\pi_i$.

11 Different Supplementary Formulas

11.1 Derivatives

$$\frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d}{dx}\exp\left[f(x)\right] = \exp\left[f(x)\right]\frac{d}{dx}f(x)$$

11.2 Integrals

11.2.1 Indefinite

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1$$
$$\int \exp(ax)dx = \frac{1}{a}\exp(ax)$$
$$\int x\exp(ax)dx = \exp(ax)\left[\frac{x}{a} - \frac{1}{a^2}\right]$$
$$\int x^2\exp(ax)dx = \exp(ax)\left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right]$$
$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

11.2.2 Definite

$$\int_0^\infty \exp(-a^2x^2)dx = \frac{\sqrt{\pi}}{2a}$$
$$\int_0^\infty x^2\exp(-a^2x^2)dx = \frac{\sqrt{\pi}}{4a^3}$$
$$\int_{-\infty}^\infty \delta(x)dx = 1$$
$$\int_{-\infty}^\infty f(x)\delta(x-a)dx = f(a)$$

11.3 Fourier Transform

11.3.1 Properties

$$\frac{d^n}{dt^n}f(t) \overset{\mathscr{F}}{\Longleftrightarrow} (j2\pi f)^n F(f)$$
$$f(-t) \overset{\mathscr{F}}{\Longleftrightarrow} F^*(f)$$
$$f(t-t_0) \overset{\mathscr{F}}{\Longleftrightarrow} F(f)e^{-j2\pi f t_0}$$
$$f(t)e^{j2\pi f_0 t} \overset{\mathscr{F}}{\Longleftrightarrow} F(f-f_0)$$

11.3.2 Transform

$$u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2}\left(\frac{1}{j\pi f} + \delta(f)\right)$$
$$\exp(-at)u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{a+j2\pi f}$$
$$t\exp(-at)u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{(a+j2\pi f)^2}$$
$$\exp(-a|t|) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{2a}{a^2+4\pi^2 f^2}$$
$$\exp(-at^2) \overset{\mathscr{F}}{\Longleftrightarrow} \sqrt{\frac{\pi}{a}}\exp\left(-\frac{(\pi f)^2}{a}\right)$$
$$\cos(2\pi f_a t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2}\big[\delta(f-f_a) + \delta(f+f_a)\big]$$
$$\sin(2\pi f_a t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2j}\big[\delta(f-f_a) - \delta(f+f_a)\big]$$

11.4 Trigonometry

$$\sin^2(\alpha) = \frac{1}{2}\left(1-\cos(2\alpha)\right)$$
$$\cos^2(\alpha) = \frac{1}{2}\left(1+\cos(2\alpha)\right)$$
$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$
$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$
$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin\left((\alpha+\beta)\right)\right]$$

11.5 Matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\quad \det[A] = ad - bc$$
$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$