המחלקה להנדסת חשמל ואלקטרוניקה

תאריך הבחינה: 23.02.17

שעות הבחינה: 9:00-12:00

מבוא לאותות אקראיים

מועד ב'

ד"ר דימה בחובסקי

תשע"ז סמסטר א'

חומר עזר - עד 4 דפי נוסחאות אישיים (משני צדדים), מחשבון

הוראות מיוחדות:

- השאלון כולל 3 שאלות ללא בחירה, סך הכל של 110 נקודות.
- יש לציין באופן מלא וברור את שלבי הפתרון. תשובה ללא הסבר לא תתקבלנה.
- אם לא מצויין אחרת, הסעיפים הם בעלי ניקוד זהה.

השאלון כולל 10 דפים (כולל דף זה)

בהצלחה!

1 תהליכים אקראיים – סטציאונריות (40 נק')

נתונים תהליכים אקראיים בלתי תלויים, WSS , $X(t)$ ו- $Y(t)$ כך ש-
$E[X(t)] = E[Y(t)] = 1, Var[X(t)] = \sigma_X^2, Var[Y(t)] = \sigma_Y^2$.
מתוכם מייצרים תהליכים חדשים: $V(t) = X(t) + aY(t), W(t) = X(t)Y(t)$.

1. האם $V(t)$ הוא WSS ?
2. האם $W(t)$ הוא WSS ?
3. חשב $R_{VW}(t, t+\tau)$ או $R_{VW}(t_1, t_2)$? האם מדובר בתהליכים סטציאונריים במשותף ?
4. חשב ערך של a עבורו $V(t), W(t)$ אורטוגונליים, בהינתן $R_x(\tau) = R_y(\tau)$.

2 תהליכים אקראיים – תכונות (40 נק')

נתון אות אקראי מהצורה

$$X(t) = A_c\Big[1 + m\sin\left(2\pi f_m t + \phi\right)\Big]\cos\left(2\pi f_c t + \theta\right)$$

כאשר קבועים $0 < A, m$,
$\phi, \theta \sim U[0, 2\pi]$ הם משתנים אקראיים בלתי תלויים.
מצא:

1. $E[X(t)]$
2. $Var[X(t)]$
3. $R_x(t, t+\tau)$ או $R_x(t_1, t_2)$. האם מדובר בתהליך סטציאונרי?
4. האות עובר דרך מערכת

$$H(f) = -j\text{sgn}(f) = \begin{cases} -j & f > 0 \\ j & f < 0 \end{cases}$$

חשב הספק P_y במוצא המערכת.

3 תהליכי Poisson (30 נק')

חלקיקים פוגעים בגלאי בממוצע אחד ל-2 שניות. חשב:

1. סיכוי ל-2 חלקיקים בדיוק בתוך שניה מסויימת.
2. סיכוי ללא יותר מחלקיק 1 בתוך שניה מסויימת.

מעוניינים למדוד קצב הפגיעת חלקיקים ממוצע ע"י הגלאי.

נניח תגובה להלם של הגלאי מהצורה $h(t) = e^{-at}u(t)[V]$, כאשר $a = 10\left[\frac{1}{sec}\right]$.

כתוצא מפיגעות חלקיקים נוצר אות מהצורה $X(t) = \sum_{i=1}^{\infty} h(t-t_i)$, כאשר t_i זה זמני פגיעת החלקיקים. תוחלת של האות המתקבל היא $E[X(t)] = 0.1[V]$ אחרי 10 דק' מדידה.

3. מהו קצב ממוצע של פגיעת החלקיקים בגלאי?

Random Processes – Formulas

1 Random Variables

1.1 Distributions

	Notation	PDF/PMF	CDF	$E[X]$	$\mathrm{Var}[X]$
Uniform	U[a,b]	$\frac{1}{b-a}, a \leqslant x \leqslant b$	$\begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leqslant x \leqslant b \\ 1 & b < x \end{cases}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$
Normal	$N(\mu, \sigma^2)$	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\Phi(x)$	μ	σ^2
Exponential	$Exp(\lambda)$	$\lambda e^{-\lambda x}, x \geqslant 0$	$1 - e^{-\lambda x}$	$1/\lambda$	$1/\lambda^2$
Poisson	$\mathcal{P}(\lambda)$	$p(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$	$e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$	λ	λ
Erlang	$Erlang(k, \lambda t)$	$\lambda \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t}$	$1 - \sum\limits_{n=0}^{k-1} \frac{(\lambda t)^n}{n!} e^{-\lambda t}$	$k/\lambda t$	$k/(\lambda t)^2$

Special properties:

- Given sum of two independent distributions $X \sim \mathcal{P}(\lambda_1)$ and $Y \sim \mathcal{P}(\lambda_2)$, the resulting distribution is given by $X + Y \sim \mathcal{P}(\lambda_1 + \lambda_2)$.
- If $X_i \sim Exp(\lambda)$ then $\sum_{i=1}^{k} X_i \sim Erlang(k, \lambda)$.

1.2 Properties

Definitions:

$$F_X(x) = p(X \leqslant x) \tag{1a}$$

$$f_X(x) = \frac{\partial F_X(x)}{\partial x} \tag{1b}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(p)\, dp \tag{1c}$$

Expectation

$$E[X] = \begin{cases} \int_{-\infty}^{\infty} x f_X(x) dx \\ \sum_i x_i p(X = x_i) \end{cases} \tag{2a}$$

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x) dx \\ \sum_i g(x_i) p(X = x_i) \end{cases} \tag{2b}$$

$$E[aX] = aE[x] \tag{2c}$$

Variance

$$\begin{aligned} \mathrm{Var}[X] &= E[(X - E[X])^2] \\ &= E[X^2] - E^2[X] \end{aligned} \tag{3a}$$

$$\mathrm{Var}[aX + b] = a^2 \,\mathrm{Var}[X] \tag{3b}$$

2 Two Random Variables

2.1 Joint Distributions

Definitions:

$$F_{XY}(x,y) = p(X \leqslant x, Y \leqslant y) \tag{4a}$$

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \tag{4b}$$

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(s,p)\, dp\, ds \tag{4c}$$

Conditional distribution:

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)},\ f_X(x) > 0 \tag{5a}$$

$$p(Y = y_j | X = x_k) = \frac{p(Y = y_j, X = x_k)}{p(X = x_k)},\ p(X = x_k) > 0 \tag{5b}$$

Expectation:

$$E[XY] = \int xy f_{XY}(x,y) dx dy \tag{6a}$$

$$E[g(X,Y)] = \int g(x,y) f_{XY}(x,y) dx dy \tag{6b}$$

$$E[X+Y] = E[X] + E[Y] \tag{6c}$$

Conditional expectation & Variance:

$$E[Y|X] = \begin{cases} \int y f_{Y|X}(y|x) dx dy \\ \sum_k y_k p(Y = y_k | X = x_j) \end{cases} \tag{7a}$$

$$E[X] = E[E[X|Y]] \tag{7b}$$

$$\mathrm{Var}[Y|X] = E[Y^2|X] - E^2[Y|X] \tag{7c}$$

$$\mathrm{Var}[Y] = \mathrm{Var}[E[Y|X]] + E[\mathrm{Var}[Y|X]] \tag{7d}$$

Independent random variables:

$$f_{XY}(x,y) = f_X(x) f_Y(y) \tag{8a}$$

$$F_{XY}(x,y) = F_X(x) F_Y(y) \tag{8b}$$

$$E[XY] = E[X]E[Y] \tag{8c}$$

$$E[g_1(X) g_2(Y)] = E[g_1(X)] E[g_2(Y)] \tag{8d}$$

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] \tag{8e}$$

Marginal distribution:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) dy \tag{9a}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y) dx \tag{9b}$$

$$F_X(x) = F_{XY}(x, \infty) \tag{9c}$$

$$F_Y(y) = F_{XY}(\infty, y) \tag{9d}$$

2.2 Correlation, Covariance & Correlation Coefficient

- For two jointly-distributed random variables X and Y, covariance is given by

  $$\begin{aligned} \mathrm{Cov}[X,Y] =& E[(X - E[X])(Y - E[Y])] \\ =& E[XY] - E[X]E[Y]. \end{aligned} \tag{10}$$

  Main covariance properties are:

  $$\mathrm{Cov}[X,X] = Var[X] \tag{11a}$$

  $$\mathrm{Cov}[X,Y] = \mathrm{Cov}[Y,X] \tag{11b}$$

  $$\mathrm{Cov}[X,a] = 0 \tag{11c}$$

  $$\mathrm{Cov}[aX,bY] = ab\,\mathrm{Cov}[X,y] \tag{11d}$$

  $$\mathrm{Cov}[X+a, Y+b] = \mathrm{Cov}[X,Y] \tag{11e}$$

  $$\mathrm{Cov}[X+Y, Z] = \mathrm{Cov}[X,Z] + \mathrm{Cov}[Y,Z] \tag{11f}$$

  $$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X,Y] \tag{11g}$$

- Correlation coefficient (also termed as Pearson product-moment correlation coefficient) is given by

  $$\rho_{XY} = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} \tag{12}$$

  such that $|\rho_{XY}| \leqslant 1$.

- For two random vectors $\mathbf{X} \in \mathbb{R}^m$ and $\mathbf{Y} \in \mathbb{R}^n$, the resulting $m \times n$ covariance matrix is given by

  $$\begin{aligned} \mathrm{Cov}[\mathbf{X},\mathbf{Y}] =& C_{\mathbf{XY}} \\ =& E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{Y} - E[\mathbf{Y}])^T] \\ =& E[\mathbf{X}\mathbf{Y}^T] - E[\mathbf{X}]E[\mathbf{Y}]^T \end{aligned} \tag{13}$$

2.3 Relations

- When X and Y are *orthogonal*, $E[XY] = 0$.
- When X and Y are *uncorrelated*, $\mathrm{Cov}[X,Y] = \rho_{XY} = 0$.
- When X and Y are *independent*, they are also uncorrelated (see also Eqs. 8).
- When X and Y are *jointly* Gaussian and uncorrelated $\Longrightarrow$ X and Y are independent.

2.4 Bi-variate & Multivariate Normal Distribution

Joint Gaussian distribution of X_1 and X_2

$$f_{X_1X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_1\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \times \right.$$
$$\left. \times \left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right]\right) \tag{14}$$

Multivariate Gaussian distribution of $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi \det\left[\mathrm{C}_{\mathbf{X}}\right]} \exp\left\{-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu}\right)^T \mathrm{C}_{\mathbf{X}}^{-1}\left(\mathbf{x}-\boldsymbol{\mu}\right)\right\}, \tag{15}$$

---

3 Signal Characterization

3.1 Auto-signal

- Average:

$$E[x(t)] = \int_{-\infty}^{\infty} x f_x(x;t)dx \tag{16}$$

- Variance:

$$Var[x(t)] = E[x^2(t)] - E^2[x(t)] \tag{17}$$

- Auto-correlation

$$R_x(t_1,t_2) = E[x(t_1)x(t_2)] \tag{18a}$$
$$R_x(t_1,t_2) = R_x(t_2,t_1) \tag{18b}$$
$$R_x(t,t) = E[x^2(t)] \tag{18c}$$

- Auto-covariance

$$C_x(t_1,t_2) = E[x(t_1)x(t_2)] - E[x(t_1)]E[x(t_2)] \tag{19a}$$
$$C_x(t,t) = Var[x(t)] \tag{19b}$$

- Correlation Coefficient

$$\rho_x(t_1,t_2) = \frac{C_x(t_1,t_2)}{\sqrt{C_x(t_1,t_1)C_x(t_2,t_2)}} \tag{20a}$$
$$|\rho_x(t_1,t_2)| \leqslant 1 \tag{20b}$$

- When $x(t_1)$ and $x(t_2)$ are *orthogonal*, $R_x(t_1,t_2) = 0$.
- When $x(t_1)$ and $x(t_2)$ are *uncorrelated*, $C_x(t_1,t_2) = \rho_x(t_1,t_2) = 0$.
- When $x(t_1)$ and $x(t_2)$ are *independent*, $R_x(t_1,t_2) = E[x(t_1)]E[x(t_2)]$.

3.2 Cross-Signal

- Cross-correlation

$$R_{xy}(t_1,t_2) = E[x(t_1)y(t_2)] \tag{21}$$

- Cross-covariance

$$C_{xy}(t_1,t_2) = R_{xy}(t_1,t_2) - E[x(t_1)]E[y(t_2)] \tag{22}$$

- Correlation Coefficient

$$\rho_{xy}(t_1,t_2) = \frac{C_{xy}(t_1,t_2)}{\sqrt{C_{xy}(t_1,t_1)C_{xy}(t_2,t_2)}} \tag{23}$$

---

4 Wide-Sense Stationary (WSS) Process

Definition:

$$E[x(t)] = E[x(0)] = m_x = \text{const} \tag{24a}$$
$$R_x(t_1,t_2) = R_x(0,\tau) = R_x(\tau), \quad \tau = t_2 - t_1, \quad \forall t_1, t_2 \tag{24b}$$

4.1 Auto-signal

- Auto-correlation

$$R_x(\tau) = E[x(t)x^*(t+\tau)] = E[x(t-\tau)x^*(t)] \tag{25}$$

Properties:

$$R_x(-\tau) = R_x(\tau) \tag{26a}$$
$$R_x(0) = E[|x(0)|^2] = E[|x(t)|^2] \tag{26b}$$
$$Var[x(t)] = C_x(0) = \sigma_x^2 \tag{26c}$$
$$|R_x(0)| \geqslant R_x(\tau) \tag{26d}$$

Deterministic definition

$$R_x(\tau) = x(\tau) * x^*(-\tau) \tag{27}$$

- Auto-covariance

$$C_x(\tau) = R_x(\tau) - m_x^2 \tag{28}$$

- Correlation Coefficient

$$\rho_x(\tau) = \frac{C_x(\tau)}{C_x(0)} \tag{29}$$

- Power spectral density (PSD)

$$S_x(f) = \mathcal{F}\{R_x(\tau)\} \tag{30}$$

Properties:

$$S_x(f) = S_x(-f) \tag{31a}$$
$$S_x(f) \geqslant 0, \ \forall f \tag{31b}$$
$$S_x(f) \in \mathbb{R} \quad \text{(real numbers)} \tag{31c}$$

Average power

$$P_x = E[x^2(t)] = R_x(0) = \int_{-\infty}^{\infty} S_x(f)df \tag{32}$$

Deterministic $X(f) = \mathcal{F}\{x(t)\}, X^*(f) = \mathcal{F}\{x^*(-\tau)\}$ definition

$$S_x(f) = X(f)X^*(f) = |X(f)|^2 \tag{33}$$

4.2 Cross-signal

- Cross-correlation

$$R_{xy}(\tau) = E[x(t)y^*(t+\tau)] \tag{34}$$

Properties

$$R_{xy}(\tau) = R_{yx}^*(-\tau) \tag{35a}$$
$$|R_{xy}(\tau)| \leqslant \sqrt{R_x(0)R_y(0)} \tag{35b}$$
$$|R_{xy}(\tau)| \leqslant \frac{1}{2}\left[R_x(0) + R_y(0)\right] \tag{35c}$$

Deterministic definition

$$R_{xy}(\tau) = x(\tau) * y^*(-\tau) \tag{36}$$

- Cross-covariance

$$C_{xy}(\tau) = R_{xy}(\tau) - m_x m_y \tag{37}$$

- Cross-PSD

$$S_{xy}(f) = \mathcal{F}\{R_{xy}(\tau)\} \tag{38}$$

Properties

$$S_{xy}(f) = S_{yx}(-f) = S_{xy}^*(-f) \tag{39}$$

Deterministic definition

$$S_{xy}(f) = X(f)Y^*(f) \tag{40}$$

4.3 White Noise Process

White noise process, $n(t)$, is WSS process that is characterized by

$$R_n(\tau) = \sigma^2\delta(\tau) \tag{41a}$$
$$S_n(f) = \sigma^2 \tag{41b}$$

5 LTI and WSS Random Process

For LTI system with impulse response $h(t)$

$$y(t) = x(t) * h(t) \tag{42}$$

Average

$$m_y = m_x \int_{-\infty}^{\infty} h(s)ds = m_x H(f=0) \tag{43}$$

5.1 Cross-correlation & cross-covariance

$$R_{xy}(\tau) = R_x(\tau) * h(\tau) \tag{44a}$$
$$C_{xy}(\tau) = C_x(\tau) * h(\tau) \tag{44b}$$
$$R_{yx}(\tau) = R_x(\tau) * h^*(-\tau) \tag{44c}$$
$$C_{yx}(\tau) = C_x(\tau) * h^*(-\tau) \tag{44d}$$
$$R_y(\tau) = R_x(\tau) * h(\tau) * h^*(-\tau) \tag{44e}$$
$$C_y(\tau) = C_x(\tau) * h(\tau) * h^*(-\tau) \tag{44f}$$

5.2 Power-Spectral Density (PSD) & Cross-PSD

Given frequency response $H(f) = \mathcal{F}\{h(\tau)\},\ H^*(f) = \mathcal{F}\{h^*(-\tau)\}$

$$S_{xy}(f) = S_x(f)\, H(f) \tag{45a}$$
$$S_{yx}(f) = S_x(f)\, H^*(f) \tag{45b}$$
$$S_y(f) = S_x(f)\, H(f)H^*(f) = S_x(f)\, |H(f)|^2 \tag{45c}$$

Power of the process:

$$P_x = R_x(0) = \int_{-\infty}^{\infty} S_x(f)\, df \tag{46a}$$

$$P_y = R_y(0) = \int_{-\infty}^{\infty} S_x(f)\, |H(f)|^2 df \tag{46b}$$

---

6 Filtering of WSS Process

6.1 SNR

For an input

$$x(t) = s(t) + n(t), \tag{47}$$

where signal $s(t)$ and noise $n(t)$ are independent and $E[n(t)] = 0$, and output $y(t)$

$$S_y(f) = S_x(f)\, |H(f)|^2 = S_s(f)\, |H(f)|^2 + S_n(f)\, |H(f)|^2\,, \tag{48}$$

where $S_s(f)\, |H(f)|^2$ is signal output PSD and $S_n(f)\, |H(f)|^2$ is noise PSD.

The input and output SNRs is given by

$$\mathrm{SNR}_x = \frac{E[s^2(t)]}{E[n^2(t)]} = \frac{R_{ss}(0)}{R_{nn}(0)} = \frac{\int S_{ss}(f)df}{\int S_{nn}(f)df} \tag{49a}$$

$$\mathrm{SNR}_y = \frac{\int S_{ss}(f\, |H(f)|^2\, df}{\int S_{nn}(f)\, |H(f)|^2\, df}. \tag{49b}$$

6.2 Match Filter

The goal of filter $h(t)$ is to provide maximum SNR at time $t = t_0$ for *deterministic* signal $x(t)$ and noise $n(t)$. The filter is given by

$$H(f) = \alpha \frac{X^*(f)}{S_n(f)} e^{-j2\pi f t_0}. \tag{50}$$

For white noise, $n(t)$, with $S_N(f) = N_0/2$, the filter is given by

$$H_{\mathrm{mf}}(f) = X^*(f)e^{-j2\pi f t_0} \longleftrightarrow h_{\mathrm{mf}}(t) = x(t_0 - t) \tag{51}$$

and the resulting maximum SNR is given by

$$SNR_{\max} = \frac{2}{N_0} \int_{-\infty}^{\infty} |X(f)|^2 \, df = \frac{2E_x}{N_0} \tag{52}$$

7 Poisson Process

The Poisson process, $N(t)$, is described by

$$p(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, \ldots \tag{53a}$$

$$p(N(0) = 0) = 0 \tag{53b}$$

Time increment property

$$p(N(t_2) - N(t_1) = k) = p(N(t_2 - t_1) = k) \tag{54}$$

7.1 Campbell Theorem

Given the relation

$$y(t) = \sum_{k=1}^{\infty} g(t - t_k) \tag{55}$$

where t_k are Poisson event times and $g(t)$ is casual system impulse response, resulting statistics is given by

$$E[y(t)] = \lambda \int_0^t g(u) du \tag{56a}$$

$$\mathrm{Var}[y(t)] = \lambda \int_0^t g^2(u) du \tag{56b}$$

7.2 Auto-correlation & auto-covariance

$$C_y(t, t + \tau) = \lambda \int_0^t h(u)h(u + \tau) du \tag{57a}$$

$$R_y(t, t + \tau) = \lambda \int_0^t h(u)h(u + \tau) du + E[y(t)]E[y(t + \tau)] \tag{57b}$$

For time-limited $h(t)$, such that $h(t) = 0, t > t_h$, or at limit $t \to \infty$

$$C_y(\tau) = C_y(t, t + \tau) \tag{58a}$$

$$R_y(\tau) = R_y(t, t + \tau) = \lambda \int_0^t h(u)h(u + \tau) du + E^2[y(t)] \tag{58b}$$

8 Different Supplementary Formulas

8.1 Derivatives

$$\frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d}{dx}\exp\left[f(x)\right] = \exp\left[f(x)\right]\frac{d}{dx}f(x)$$

8.2 Integrals

8.2.1 Indefinite

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1$$
$$\int \exp(ax)dx = \frac{1}{a}\exp(ax)$$
$$\int x\exp(ax)dx = \exp(ax)\left[\frac{x}{a} - \frac{1}{a^2}\right]$$
$$\int x^2\exp(ax)dx = \exp(ax)\left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right]$$
$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

8.2.2 Definite

$$\int_0^\infty \exp(-a^2x^2)dx = \frac{\sqrt{\pi}}{2a}$$
$$\int_0^\infty x^2\exp(-a^2x^2)dx = \frac{\sqrt{\pi}}{4a^3}$$
$$\int_{-\infty}^\infty \delta(x)dx = 1$$
$$\int_{-\infty}^\infty f(x)\delta(x-a)dx = f(a)$$

8.3 Fourier Transform

8.3.1 Properties

$$\frac{d^n}{dt^n}f(t) \overset{\mathscr{F}}{\Longleftrightarrow} (j2\pi f)^n F(f)$$

8.3.2 Transform

$$u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2}\left(\frac{1}{j\pi f} + \delta(f)\right)$$
$$\exp(-at)u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{a+j2\pi f}$$
$$t\exp(-at)u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{(a+j2\pi f)^2}$$
$$\exp(-a|t|) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{2a}{a^2+4\pi^2f^2}$$
$$\exp(-at^2) \overset{\mathscr{F}}{\Longleftrightarrow} \sqrt{\frac{\pi}{a}}\exp\left(-\frac{(\pi f)^2}{a}\right)$$
$$\cos(2\pi f_a t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2}\big[\delta(f-f_a) + \delta(f+f_a)\big]$$
$$\sin(2\pi f_a t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2j}\big[\delta(f-f_a) - \delta(f+f_a)\big]$$

8.4 Trigonometry

$$\sin^2(\alpha) = \frac{1}{2}\left(1-\cos(2\alpha)\right)$$
$$\cos^2(\alpha) = \frac{1}{2}\left(1+\cos(2\alpha)\right)$$
$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$
$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$
$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin\left((\alpha+\beta)\right)\right]$$