המחלקה להנדסת חשמל ואלקטרוניקה

תאריך הבחינה: 1.2.17
שעות הבחינה: 13:30-16:30

# מבוא לאותות אקראיים

## מועד א'

ד"ר דימה בחובסקי

תשע"ז סמסטר א'

**חומר עזר** - עד 4 דפי נוסחאות אישיים (משני צדדים), מחשבון

הוראות מיוחדות:

- השאלון כולל 3 שאלות ללא בחירה, סך הכל של 110 נקודות.
- יש לציין באופן מלא וברור את שלבי הפתרון. תשובה ללא הסבר לא תתקבלנה.
- אם לא מצויין אחרת, הסעיפים הם בעלי ניקוד זהה.

השאלון כולל 9 דפים (כולל דף זה)

בהצלחה!

## 1 מסנן מתואם (30 נק')

נתון אות דטרמיניסטי $x(t)=u(t)\left[\exp\left(-\alpha t\right)-\exp\left(-\alpha\beta t\right)\right]$ כאשר $\alpha,\beta>0$ ו- $u(t)$ הוא אות מדרגה.

לאות נוסף רעש לבן גאוסי בעל צפיפות הספק ספקטראלית $S_n(f)=N_0/2$.

1. מהו מסנן המתואם עבור האות, $h(t)$ ?
2. מהו הספק האות במוצא המסנן בזמן $t_0$ ?
3. מהו הספק הרעש במוצא המסנן ? מהו SNR המתקבל?

## 2 תכונות ומאפיינים של תהליכים אקראיים (50 נק')

נתונים תהליכים אקראיים : $X(t)=At^2, Y(t)=A^2t^2$,

כאשר A הינו משתנה אקראי המתפלג אחיד, $A\sim U[0,1]$.

חשב :

1. $E[X(t)]$
2. $R_x(t_1,t_2)$,

   האם התהליך $X(t)$ הוא סטציואנרי?
3. $E[Y(t)]$
4. $R_{xy}(t,t+\tau), C_{xy}(t,t+\tau)$

   האם תהליכים $X(t),Y(t)$ הם סטציאונריים במשותף, $R_{xy}(t,t+\tau)\overset{?}{=}R_{xy}(\tau)$ ?
5. $p(Y(t)>3|X(t)=2)$

## 3 תהליכים סטציאונריים – גוזר (30 נק')

נתון גוזר, $Y(t)=\dfrac{d}{dt}X(t)$.

הכניסה היא תהליך אקראי גאוסי בעל $E[X(t)]=0, R_x(\tau)=\sigma_x^2e^{-\alpha^2\tau^2}$.

חשב :

1. $R_{xy}(t,t+\tau)$

האם תהליכים $X(t),Y(t)$ הם סטציאונריים במשותף, $R_{xy}(t,t+\tau)\overset{?}{=}R_{xy}(\tau)$ ?

2. הספק $P_y$.

3. בהנתן פילוג של $X(t)\sim N(0,\sigma_x^2)$, מהו צפיפות הפילוג (PDF) של האות $Y(t)$ ? פרט!

# 4 נוסחאות שימושיות

## 4.1 נגזרת

$$\frac{d}{dx}x^n = nx^{n-1} \tag{1}$$

$$\frac{d}{dx}\exp(x) = \exp(x) \tag{2}$$

$$\frac{d}{dx}\exp(f(x)) = f'(x)\exp(ax) \tag{3}$$

## 4.2 אינטגרל

### 4.2.1 לא מסויים

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n\neq -1 \tag{4}$$

$$\int \exp(-ax)dx = \frac{1}{a}\exp(-ax) \tag{5}$$

$$\int x\exp(-ax)dx = \exp(-ax)\left[\frac{x}{a}-\frac{1}{a^2}\right] \tag{6}$$

(7)

$$\int x^2\exp(-ax)dx = \exp(-ax)\left[\frac{x^2}{a}-\frac{2x}{a^2}-\frac{2}{a^3}\right]$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} \tag{8}$$

### 4.2.2 מסויים

$$\int_0^\infty \exp(-a^2x^2)dx = \frac{\sqrt{\pi}}{2a} \tag{9}$$

$$\int_0^\infty x^2\exp(-a^2x^2)dx = \frac{\sqrt{\pi}}{4a^3} \tag{10}$$

## 4.3 התמרות פוריה

### 4.3.1 תכונות

$$\frac{d^n}{dt^n}f(t) \overset{\mathscr{F}}{\Longleftrightarrow} (j2\pi f)^n F(f) \tag{11}$$

(12)

### 4.3.2 התמרות

$$u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2}\left(\frac{1}{j\pi f}+\delta(f)\right) \tag{13}$$

$$\exp(-at)u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{a+j2\pi f} \tag{14}$$

$$t\exp(-at)u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{(a+j2\pi f)^2} \tag{15}$$

$$\exp(-a|t|) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{2a}{a^2+4\pi^2f^2} \tag{16}$$

$$\exp(-at^2) \overset{\mathscr{F}}{\Longleftrightarrow} \sqrt{\frac{\pi}{a}}\exp\left(-\frac{(\pi f)^2}{a}\right) \tag{17}$$

# Random Processes – Formulas

## 1 Random Variables

### 1.1 Distributions

| | Notation | PDF/PMF | CDF | $E[X]$ | Var$[X]$ |
|---|---|---|---|---|---|
| Uniform | U[a,b] | $\frac{1}{b-a}, a \leqslant x \leqslant b$ | $\begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leqslant x \leqslant b \\ 1 & b < x \end{cases}$ | $\dfrac{a+b}{2}$ | $\dfrac{(b-a)^2}{12}$ |
| Normal | $N(\mu, \sigma^2)$ | $\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ | $\Phi(x)$ | $\mu$ | $\sigma^2$ |
| Exponential | $Exp(\lambda)$ | $\lambda e^{-\lambda x}, x \geqslant 0$ | $1 - e^{-\lambda x}$ | $1/\lambda$ | $1/\lambda^2$ |
| Poisson | $\mathcal{P}(\lambda)$ | $p(X = k) = \lambda^k \frac{e^{-\lambda}}{k!}$ | $e^{-\lambda} \sum_{i=0}^{k} \frac{\lambda^i}{i!}$ | $\lambda$ | $\lambda$ |
| Erlang | $Erlang(k, \lambda t)$ | $\lambda \frac{(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t}$ | $1 - \sum\limits_{n=0}^{k-1} \frac{(\lambda t)^n}{n!} e^{-\lambda t}$ | $k/\lambda t$ | $k/(\lambda t)^2$ |

Special properties:

- Given sum of two independent distributions $X \sim \mathcal{P}(\lambda_1)$ and $Y \sim \mathcal{P}(\lambda_2)$, the resulting distribution is given by $X + Y \sim \mathcal{P}(\lambda_1 + \lambda_2)$.
- If $X_i \sim Exp(\lambda)$ then $\sum_{i=1}^{k} X_i \sim Erlang(k, \lambda)$.

### 1.2 Properties

Definitions:

$$F_X(x) = p(X \leqslant x) \tag{1a}$$

$$f_X(x) = \frac{\partial F_X(x)}{\partial x} \tag{1b}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(p)\, dp \tag{1c}$$

Expectation

$$E[X] = \begin{cases} \int_{-\infty}^{\infty} x f_X(x) dx \\ \sum_i x_i p(X = x_i) \end{cases} \tag{2a}$$

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x) dx \\ \sum_i g(x_i) p(X = x_i) \end{cases} \tag{2b}$$

$$E[aX] = aE[x] \tag{2c}$$

Variance

$$\begin{aligned} \mathrm{Var}[X] =& E[(X - E[X])^2] \\ =& E[X^2] - E^2[X] \end{aligned} \tag{3a}$$

$$\mathrm{Var}[aX + b] = a^2 \,\mathrm{Var}[X] \tag{3b}$$

# 2 Two Random Variables

## 2.1 Joint Distributions

Definitions:

$$F_{XY}(x,y) = p(X \leqslant x, Y \leqslant y) \tag{4a}$$

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y} \tag{4b}$$

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(s,p)\,dp\,ds \tag{4c}$$

Conditional distribution:

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)},\ f_X(x) > 0 \tag{5a}$$

$$p(Y = y_j | X = x_k) = \frac{p(Y = y_j, X = x_k)}{p(X = x_k)},\ p(X = x_k) > 0 \tag{5b}$$

Expectation:

$$E[XY] = \int xy f_{XY}(x,y) dx dy \tag{6a}$$

$$E[g(X,Y)] = \int g(x,y) f_{XY}(x,y) dx dy \tag{6b}$$

$$E[X+Y] = E[X] + E[Y] \tag{6c}$$

Conditional expectation & Variance:

$$E[Y|X] = \begin{cases} \int y f_{Y|X}(y|x) dx dy \\ \sum_k y_k p(Y = y_k | X = x_j) \end{cases} \tag{7a}$$

$$E[X] = E[E[X|Y]] \tag{7b}$$

$$\text{Var}[Y|X] = E[Y^2|X] - E^2[Y|X] \tag{7c}$$

$$\text{Var}[Y] = \text{Var}[E[Y|X]] + E[\text{Var}[Y|X]] \tag{7d}$$

Independent random variables:

$$f_{XY}(x,y) = f_X(x) f_Y(y) \tag{8a}$$

$$F_{XY}(x,y) = F_X(x) F_Y(y) \tag{8b}$$

$$E[XY] = E[X]E[Y] \tag{8c}$$

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] \tag{8d}$$

Marginal distribution:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y) dy \tag{9a}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y) dx \tag{9b}$$

$$F_X(x) = F_{XY}(x,\infty) \tag{9c}$$

$$F_Y(y) = F_{XY}(\infty,y) \tag{9d}$$

## 2.2 Correlation, Covariance & Correlation Coefficient

- For two jointly-distributed random variables $X$ and $Y$, covariance is given by

$$\begin{aligned}\text{Cov}[X,Y] &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY] - E[X]E[Y].\end{aligned} \tag{10}$$

  Main covariance properties are:

$$\text{Cov}[X,X] = Var[X] \tag{11a}$$

$$\text{Cov}[X,Y] = \text{Cov}[Y,X] \tag{11b}$$

$$\text{Cov}[X,a] = 0 \tag{11c}$$

$$\text{Cov}[aX,bY] = ab\,\text{Cov}[X,y] \tag{11d}$$

$$\text{Cov}[X+a,Y+b] = \text{Cov}[X,Y] \tag{11e}$$

$$\text{Cov}[X+Y,Z] = \text{Cov}[X,Z] + \text{Cov}[Y,Z] \tag{11f}$$

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] \tag{11g}$$

- Correlation coefficient (also termed as Pearson product-moment correlation coefficient) is given by

$$\rho_{XY} = \frac{\text{Cov}[X,Y]}{\sqrt{\text{Var}[X]\,\text{Var}[Y]}} \tag{12}$$

  such that $|\rho_{XY}| \leqslant 1$.

- For two random vectors $\mathbf{X} \in \mathbb{R}^m$ and $\mathbf{Y} \in \mathbb{R}^n$, the resulting $m \times n$ covariance matrix is given by

$$\begin{aligned}\text{Cov}[\mathbf{X},\mathbf{Y}] &= C_{\mathbf{XY}} \\ &= E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{Y} - E[\mathbf{Y}])^T] \\ &= E[\mathbf{X}\mathbf{Y}^T] - E[\mathbf{X}]E[\mathbf{Y}]^T\end{aligned} \tag{13}$$

## 2.3 Relations

- When $X$ and $Y$ are *orthogonal*, $E[XY] = 0$.
- When $X$ and $Y$ are *uncorrelated*, $\text{Cov}[X,Y] = \rho_{XY} = 0$.
- When $X$ and $Y$ are *independent*, they are also uncorrelated (see also Eqs. 8).
- When $X$ and $Y$ are *jointly* Gaussian and uncorrelated $\Longrightarrow$ $X$ and $Y$ are independent.

### 2.4 Bi-variate & Multivariate Normal Distribution

Joint Gaussian distribution of $X_1$ and $X_2$

$$f_{X_1X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_1\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\times \right.$$
$$\left. \times\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right]\right) \quad (14)$$

Multivariate Gaussian distribution of $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi \det\left[\mathrm{C}_{\mathbf{X}}\right]} \exp\left\{-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu}\right)^T \mathrm{C}_{\mathbf{X}}^{-1}\left(\mathbf{x}-\boldsymbol{\mu}\right)\right\}, \quad (15)$$

## 3 Signal Characterization

### 3.1 Auto-signal

- Average:

$$E[x(t)] = \int_{-\infty}^{\infty} x f_x(x;t)dx \quad (16)$$

- Variance:

$$Var[x(t)] = E[x^2(t)] + E^2[x(t)] \quad (17)$$

- Auto-correlation

$$R_x(t_1,t_2) = E[x(t_1)x(t_2)] \quad (18a)$$
$$R_x(t_1,t_2) = R_x(t_2,t_1) \quad (18b)$$
$$R_x(t,t) = E[x^2(t)] \quad (18c)$$

- Auto-covariance

$$C_x(t_1,t_2) = E[x(t_1)x(t_2)] - E[x(t_1)]E[x(t_2)] \quad (19a)$$
$$C_x(t,t) = Var[x(t)] \quad (19b)$$

- Correlation Coefficient

$$\rho_x(t_1,t_2) = \frac{C_x(t_1,t_2)}{\sqrt{C_x(t_1,t_1)C_x(t_2,t_2)}} \quad (20a)$$
$$|\rho_x(t_1,t_2)| \leqslant 1 \quad (20b)$$

- When $x(t_1)$ and $x(t_2)$ are *orthogonal*, $R_x(t_1,t_2) = 0$.
- When $x(t_1)$ and $x(t_2)$ are *uncorrelated*, $C_x(t_1,t_2) = \rho_x(t_1,t_2) = 0$.
- When $x(t_1)$ and $x(t_2)$ are *independent*, $R_x(t_1,t_2) = E[x(t_1)]E[x(t_2)]$.

### 3.2 Cross-Signal

- Cross-correlation

$$R_{xy}(t_1,t_2) = E[x(t_1)y(t_2)] \quad (21)$$

- Cross-covariance

$$C_{xy}(t_1,t_2) = R_{xy}(t_1,t_2) - E[x(t_1)]E[y(t_2)] \quad (22)$$

- Correlation Coefficient

$$\rho_{xy}(t_1,t_2) = \frac{C_{xy}(t_1,t_2)}{\sqrt{C_{xy}(t_1,t_1)C_{xy}(t_2,t_2)}} \quad (23)$$

## 4 Wide-Sense Stationary (WSS) Process

Definition:

$$E[x(t)] = E[x(0)] = m_x = \text{const} \quad (24a)$$
$$R_x(t_1,t_2) = R_x(0,\tau) = R_x(\tau), \quad \tau = t_2 - t_1, \quad \forall t_1, t_2 \quad (24b)$$

### 4.1 Auto-signal

- Auto-correlation

$$R_x(\tau) = E[x(t)x^*(t+\tau)] = E[x(t-\tau)x^*(t)] \tag{25}$$

Properties:

$$R_x(-\tau) = R_x(\tau) \tag{26a}$$
$$R_x(0) = E[|x(0)|^2] = E[|x(t)|^2] \tag{26b}$$
$$Var[x(t)] = C_x(0) = \sigma_x^2 \tag{26c}$$
$$|R_x(0)| \geqslant R_x(\tau) \tag{26d}$$

Deterministic definition

$$R_x(\tau) = x(\tau) * x^*(-\tau) \tag{27}$$

- Auto-covariance

$$C_x(\tau) = R_x(\tau) - m_x^2 \tag{28}$$

- Correlation Coefficient

$$\rho_x(\tau) = \frac{C_x(\tau)}{C_x(0)} \tag{29}$$

- Power spectral density (PSD)

$$S_x(f) = \mathcal{F}\{R_x(\tau)\} \tag{30}$$

Properties:

$$S_x(f) = S_x(-f) \tag{31a}$$
$$S_x(f) \geqslant 0, \ \forall f \tag{31b}$$
$$S_x(f) \in \mathbb{R} \quad \text{(real numbers)} \tag{31c}$$

Average power

$$P_x = E[x^2(t)] = R_x(0) = \int_{-\infty}^{\infty} S_x(f)df \tag{32}$$

Deterministic $X(f) = \mathcal{F}\{x(t)\}, X^*(f) = \mathcal{F}\{x^*(-\tau)\}$ definition

$$S_x(f) = X(f)X^*(f) = |X(f)|^2 \tag{33}$$

### 4.2 Cross-signal

- Cross-correlation

$$R_{xy}(\tau) = E[x(t)y^*(t+\tau)] \tag{34}$$

Properties

$$R_{xy}(\tau) = R_{yx}^*(-\tau) \tag{35a}$$
$$|R_{xy}(\tau)| \leqslant \sqrt{R_x(0)R_y(0)} \tag{35b}$$
$$|R_{xy}(\tau)| \leqslant \frac{1}{2}\left[R_x(0) + R_y(0)\right] \tag{35c}$$

Deterministic definition

$$R_{xy}(\tau) = x(\tau) * y^*(-\tau) \tag{36}$$

- Cross-covariance

$$C_{xy}(\tau) = R_{xy}(\tau) - m_x m_y \tag{37}$$

- Cross-PSD

$$S_{xy}(f) = \mathcal{F}\{R_{xy}(\tau)\} \tag{38}$$

Properties

$$S_{xy}(f) = S_{yx}(-f) = S_{xy}^*(-f) \tag{39}$$

Deterministic definition

$$S_{xy}(f) = X(f)Y^*(f) \tag{40}$$

### 4.3 White Noise Process

White noise process, $n(t)$, is WSS process that is characterized by

$$R_n(\tau) = \sigma^2\delta(\tau) \tag{41a}$$
$$S_n(f) = \sigma^2 \tag{41b}$$

# 5 LTI and WSS Random Process

For LTI system with impulse response $h(t)$

$$y(t) = x(t) * h(t) \tag{42}$$

Average

$$m_y = m_x \int_{-\infty}^{\infty} h(s)ds = m_x H(f=0) \tag{43}$$

## 5.1 Cross-correlation & cross-covariance

$$R_{xy}(\tau) = R_x(\tau) * h(\tau) \tag{44a}$$
$$C_{xy}(\tau) = C_x(\tau) * h(\tau) \tag{44b}$$
$$R_{yx}(\tau) = R_x(\tau) * h^*(-\tau) \tag{44c}$$
$$C_{yx}(\tau) = C_x(\tau) * h^*(-\tau) \tag{44d}$$
$$R_y(\tau) = R_x(\tau) * h(\tau) * h^*(-\tau) \tag{44e}$$
$$C_y(\tau) = C_x(\tau) * h(\tau) * h^*(-\tau) \tag{44f}$$

## 5.2 Power-Spectral Density (PSD) & Cross-PSD

Given frequency response $H(f) = \mathcal{F}\{h(\tau)\}$, $H^*(f) = \mathcal{F}\{h^*(-\tau)\}$

$$S_{xy}(f) = S_x(f) H(f) \tag{45a}$$
$$S_{yx}(f) = S_x(f) H^*(f) \tag{45b}$$
$$S_{yy}(f) = S_x(f) H(f) H^*(f) = S_x(f) |H(f)|^2 \tag{45c}$$

Power of the process:

$$P_x = R_x(0) = \int_{-\infty}^{\infty} S_x(f)\, df \tag{46a}$$

$$P_y = R_y(0) = \int_{-\infty}^{\infty} S_x(f) |H(f)|^2 df \tag{46b}$$

# 6 Filtering of WSS Process

## 6.1 SNR

For an input

$$x(t) = s(t) + n(t), \tag{47}$$

where signal $s(t)$ and noise $n(t)$ are independent and $E[n(t)] = 0$, and output $y(t)$

$$S_y(f) = S_x(f) |H(f)|^2 = S_s(f) |H(f)|^2 + S_n(f) |H(f)|^2, \tag{48}$$

where $S_s(f) |H(f)|^2$ is signal output PSD and $S_n(f) |H(f)|^2$ is noise PSD.

The input and output SNRs is given by

$$\mathrm{SNR}_x = \frac{E[s^2(t)]}{E[n^2(t)]} = \frac{R_{ss}(0)}{R_{nn}(0)} = \frac{\int S_{ss}(f)df}{\int S_{nn}(f)df} \tag{49a}$$

$$\mathrm{SNR}_y = \frac{\int S_{ss}(f |H(f)|^2 df}{\int S_{nn}(f) |H(f)|^2 df}. \tag{49b}$$

## 6.2 Match Filter

The goal of filter $h(t)$ is to provide maximum SNR at time $t = t_0$ for *deterministic* signal $x(t)$ and noise $n(t)$. For white noise, $n(t)$, with $S_N(f) = N_0/2$, the filter is given by

$$H_{\mathrm{mf}}(f) = X^*(f) e^{-j2\pi f t_0} \longleftrightarrow h_{\mathrm{mf}}(t) = x(t_0 - t) \tag{50}$$

and the resulting maximum SNR is given by

$$SNR_{\max} = \frac{2}{N_0} \int_{-\infty}^{\infty} |X(f)|^2\, df = \frac{2E_x}{N_0} \tag{51}$$

# 7 Poisson Process

The Poisson process, $N(t)$, is described by

$$p(N(t) = k) = e^{\lambda t}\frac{(\lambda t)^k}{k!}, \quad k = 0, 1, \dots \tag{52}$$

Time increment property

$$N(t_2) - N(t_1) = N(t_2 - t_1) \tag{53}$$

## 7.1 Campbell Theorem

Given the relation

$$y(t) = \sum_{k=1}^{\infty} g(t - t_k) \tag{54}$$

where $t_k$ are Poisson event times and $g(t)$ is casual system impulse response, resulting statistics is given by

$$E[y] = \lambda \int_0^{\infty} g(u)du \tag{55a}$$

$$\mathrm{Var}[y] = \lambda^2 \int_0^{\infty} g^2(u)du \tag{55b}$$