Random Processes – Formulas

1 Distributions

1.1 Continuous

	Notation	PDF	CDF	$E[X]$	$\mathrm{Var}[X]$
Uniform	U[a,b]	$\begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & b < x \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal	$N(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$	$\Phi(x)$	μ	σ^2
Exponential	$Exp(\lambda)$	$\lambda \exp(-\lambda x), x \ge 0$	$1 - \exp(-\lambda x)$	$1/\lambda$	$1/\lambda^2$

1.1.1 Q-function

Given $Y \sim N(\mu, \sigma^2)$

$$\frac{Y-\mu}{\sigma} \sim N(0,1) \tag{1}$$

$$p(Y > y) = Q\left(\frac{y-\mu}{\sigma}\right) \tag{2}$$

$$Q(x) = 1 - \Phi(x) \tag{3}$$

$$Q(-x) = 1 - Q(x) \tag{4}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{s^2}{2}\right) ds \tag{5}$$

1.2 Discrete

	Notation	PDF	CDF	$E[X]$	$\mathrm{Var}[X]$
Bernoulli	Ber(p)	$\begin{cases} 1-p & k=0 \\ p & k=1 \end{cases}$	$\begin{cases} 0 & x < 0 \\ 1-p & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$	p	$p(1-p)$
Binomial	Bin(n,p)	$\binom{n}{k} p^k (1-p)^{n-k}$		np	$np(1-p)$
Geometric	Geo(p)	$p(1-p)^{k-1}$	$1-(1-p)^k$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

2 Random Variables

Definitions:

$$F_X(x) = p(X \le x) \tag{6}$$

$$f_X(x) = \frac{\partial F_X(x)}{\partial x} \ge 0 \tag{7}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(p)\, dp \tag{8}$$

$$p(a < X \leqslant b) = F_X(b) - F_X(a) \tag{9}$$

$$f_X(x) \ge 0 \tag{10}$$

$$\int_{-\infty}^{\infty} f_X(x) dx = 1 \tag{11}$$

$$p_X[x_k] = \Pr[X = x_k] \tag{12}$$

$$0 \le p_X[x_i] \le 1 \ \forall i \tag{13}$$

$$\sum_i p_X[x_i] = 1 \tag{14}$$

$$F_X(x) = \Pr(X \leq x),\ x \in \mathbb{R} \tag{15}$$

$$F_X(x) = \sum_{k:x_k \leqslant x} p_X[x_k] \tag{16}$$

Expectation:

$$E[X] = \begin{cases} \int_{-\infty}^{\infty} x f_X(x) dx \\ \sum_i x_i p_X[x_i] \end{cases} \tag{17a}$$

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x) dx \\ \sum_i g(x_i) p_X[x_i] \end{cases} \tag{17b}$$

$$E[aX + b] = aE[x] + b \tag{17c}$$

Variance:

$$\begin{aligned} \mathrm{Var}[X] =& E\Big[(X - E[X])^2\Big] \\ =& E[X^2] - E^2[X] \end{aligned} \tag{18a}$$

$$\mathrm{Var}[aX + b] = a^2 \mathrm{Var}[X] \tag{18b}$$

$$\mathrm{Var}[b] = 0 \tag{18c}$$

Median: Value of m, such that $\Pr(X \leq m) \geq \frac{1}{2}$ and $\Pr(X \geq x) \geq \frac{1}{2}$

2.1 Numerical calculation

$$E[X] = \frac{1}{N} \sum_{i=1}^{N} x_i \tag{19}$$

$$\mathrm{Var}[X] = \frac{1}{N} \sum_{i=1}^{N} \big(x_i - E[X]\big)^2 \tag{20}$$

2.2 Histogram

$$p_X[x_i] \approx \frac{n_i}{N} \quad i = 1, \dots, k \tag{21}$$

$$f_X(x_i) \approx \frac{n_i}{N} \cdot \frac{1}{\Delta x} \quad i = 1, \dots, k \tag{22}$$

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3 Two Random Variables

3.1 Joint Distributions

Definitions:

$$F_{XY}(x, y) = p(X \leq x, Y \leq y) \tag{23a}$$

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y} \geqslant 0 \tag{23b}$$

$$F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(s, p)\, dp\, ds \tag{23c}$$

$$p[x_j, y_k] = p(X = x_j, Y = y_k) \tag{24a}$$

$$F_{XY}(x, y) = p(X \leqslant x_j, Y \leqslant y_k) \tag{24b}$$

Expectation:

$$E[g(X, Y)] = \begin{cases} \iint g(x, y) f_{XY}(x, y) dx dy \\ \sum_i \sum_k g(x_i, y_k) p_{XY}[x_i, y_k] \end{cases} \tag{25a}$$

$$E[aX + bY] = aE[X] + bE[Y] \tag{25b}$$

For **independent** random variables:

$$f_{XY}(x, y) = f_X(x) f_Y(y) \tag{26a}$$

$$p_{XY}[x_k, y_j] = p_X[x_k] p_Y[y_j] \tag{26b}$$

$$F_{XY}(x, y) = F_X(x) F_Y(y) \tag{26c}$$

(26d)

Independent random variables properties:

$$E[XY] = E[X]E[Y] \tag{27a}$$

$$E[g_1(X) g_2(Y)] = E[g_1(X)] E[g_2(Y)] \tag{27b}$$

$$\mathrm{Var}[aX + bY] = a^2 \mathrm{Var}[X] + b^2 \mathrm{Var}[Y] \tag{27c}$$

Marginal distribution:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy \tag{28a}$$

$$p_X[x_k] = \sum_j p_{XY}[x_k, y_j] \tag{28b}$$

$$F_X(x) = F_{XY}(x, \infty) \tag{28c}$$

$$F_Y(y) = F_{XY}(\infty, y) \tag{28d}$$

3.2 Correlation, Covariance & Correlation Coefficient

- For two jointly-distributed random variables X and Y, covariance is given by

$$\begin{aligned} \mathrm{Cov}[X, Y] =& E\Big[(X - E[X])(Y - E[Y])\Big] \\ =& E[XY] - E[X]E[Y] \end{aligned} \tag{29}$$

Main covariance properties are:

$$\mathrm{Cov}[X, X] = Var[X] \tag{30a}$$

$$\mathrm{Cov}[X, Y] = \mathrm{Cov}[Y, X] \tag{30b}$$

$$\mathrm{Cov}[X,a] = 0 \tag{30c}$$
$$\mathrm{Cov}[aX,bY] = ab\,\mathrm{Cov}[X,Y] \tag{30d}$$
$$\mathrm{Cov}[X,Y] = \mathrm{Cov}[X+a,Y+b] \tag{30e}$$
$$\mathrm{Var}[X \pm Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] \pm 2\,\mathrm{Cov}[X,Y] \tag{30f}$$

- Correlation coefficient (also termed as Pearson product-moment correlation coefficient) is given by

$$\rho_{XY} = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} \tag{31}$$

such that $|\rho_{XY}| \leq 1$.

3.3 MMSE Linear Prediction

Mean square error (MSE) of predictor $\hat{Y}$ is given by

$$mse = E[(Y-\hat{Y})^2] \tag{32}$$

Linear prediction of $\hat{Y} = ax+b$ for $X=x$ is

$$\hat{Y} = E[Y] + \frac{\mathrm{Cov}[X,Y]}{\mathrm{Var}[X]}\big(x - E[X]\big) \tag{33}$$

and

$$mse_{min} = E\left[\Big(Y-(aX+b)\Big)^2\right] = \mathrm{Var}[Y]\,(1-\rho_{XY}^2) \tag{34}$$

When X, Y are jointly Gaussian, this prediction is optimal among **all** possible predictors

3.4 Relations

- When X and Y are *orthogonal*, $E[XY]=0$.
- When X and Y are *uncorrelated*, $\mathrm{Cov}[X,Y] = \rho_{XY} = 0$.
- When X and Y are *independent*, they are also uncorrelated (see also Eqs. 26).
- When X and Y are *jointly* Gaussian and uncorrelated $\Rightarrow$ X and Y are independent.

3.5 Bi-variate Normal Distribution

Joint Gaussian distribution of X_1 and X_2

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \mathrm{C}_{\mathbf{X}}\right) \tag{35}$$

with covariance matrix

$$\mathrm{C}_{\mathbf{X}} = \begin{bmatrix} \mathrm{Cov}[X_1,X_1] & \mathrm{Cov}[X_1,X_2] \\ \mathrm{Cov}[X_2,X_1] & \mathrm{Cov}[X_2,X_2] \end{bmatrix} \tag{36}$$

Important properties:

- Sum of independent Gaussian variables is a Gaussian variable.
- Random vector $[X_1,\ldots,X_n]$ is **jointly** Gaussian distributed, iff (if and only if) for all possible real vectors $\mathbf{a} = (a_1,\ldots,a_n)^T$ linear combination $Y = a_1X_1 + \cdot + a_nX_n$ is Gaussian distributed.
- If jointly distributed Gaussian random variables are *uncorrelated*, they are also *independent*

4 Random Processes – General Properties

- PDF & CDF

$$F_{\mathbf{x}}(x;t) = p(\mathbf{x}(t) \leq x) \tag{37a}$$
$$f_{\mathbf{x}}(x;t) = \frac{\partial}{\partial x}F_{\mathbf{x}}(x;t) \tag{37b}$$
$$p_{\mathbf{x}}[x_k;n] = p(\mathbf{x}[n] = x_k) \tag{37c}$$

- Average:

$$E[\mathbf{x}(t)] = \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x;t)dx \tag{38a}$$
$$E\Big[\mathbf{x}[n]\Big] = \sum_i x_i p_{\mathbf{x}}[x_k;n] \tag{38b}$$

- Variance:

$$\mathrm{Var}\big[\mathbf{x}(t)\big] = E\Big[\mathbf{x}^2(t)\Big] - E^2\big[\mathbf{x}(t)\big] = \sigma_{\mathbf{x}}^2(t) \tag{39a}$$
$$\mathrm{Var}\big[\mathbf{x}[n]\big] = E\Big[\mathbf{x}^2[n]\Big] - E^2\big[\mathbf{x}[n]\big] = \sigma_{\mathbf{x}}^2[n] \tag{39b}$$

- Auto-correlation

$$R_{\mathbf{x}}(t_1,t_2) = E[\mathbf{x}(t_1)\mathbf{x}(t_2)] \tag{40a}$$
$$R_{\mathbf{x}}(t,t+\tau) = E[\mathbf{x}(t)\mathbf{x}(t+\tau)] \tag{40b}$$

$$R_{\mathbf{x}}(t_1,t_2) = R_{\mathbf{x}}(t_2,t_1) \quad (40c)$$
$$R_{\mathbf{x}}(t,t) = E[\mathbf{x}^2(t)] \quad (40d)$$
$$R_{\mathbf{x}}[n_1,n_2] = E\left[\mathbf{x}[n_1]\mathbf{x}[n_2]\right] \quad (40e)$$
$$R_{\mathbf{x}}[n,n] = E\left[\mathbf{x}^2[n]\right] \quad (40f)$$

- Auto-covariance

$$C_{\mathbf{x}}(t_1,t_2) = E\left[\left\{\mathbf{x}(t_1) - E[\mathbf{x}(t_1)]\right\}\left\{\mathbf{x}(t_2) - E[\mathbf{x}(t_2)]\right\}\right] \quad (41)$$
$$= R_{\mathbf{x}}(t_1,t_2) - E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)] \quad (42)$$
$$C_{\mathbf{x}}[n_1,n_2] = E\left[\left\{\mathbf{x}[n_1] - E[\mathbf{x}[n_1]]\right\}\left\{\mathbf{x}[n_2] - E[\mathbf{x}[n_2]]\right\}\right] \quad (43)$$
$$= R_{\mathbf{x}}[n_1,n_2] - E\left[\mathbf{x}[n_1]\right] E\left[\mathbf{x}[n_2]\right] \quad (44)$$

$$C_{\mathbf{x}}(t,t) = \mathrm{Var}\left[\mathbf{x}(t)\right] \quad (45a)$$
$$C_{\mathbf{x}}[n,n] = \mathrm{Var}\left[\mathbf{x}[n]\right] \quad (45b)$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(t_1,t_2) = \frac{C_{\mathbf{x}}(t_1,t_2)}{\sqrt{C_{\mathbf{x}}(t_1,t_1)C_{\mathbf{x}}(t_2,t_2)}} \quad (46a)$$
$$\left|\rho_{\mathbf{x}}(t_1,t_2)\right| \leqslant 1 \quad (46b)$$

  - When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *orthogonal*, $R_{\mathbf{x}}(t_1,t_2) = 0$.
  - When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *uncorrelated*, $C_{\mathbf{x}}(t_1,t_2) = \rho_{\mathbf{x}}(t_1,t_2) = 0$.
  - When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *independent*, $R_{\mathbf{x}}(t_1,t_2) = E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)]$.

- Power

$$P_{\mathbf{x}}(t) = E\left[\mathbf{x}^2(t)\right] \quad (47a)$$
$$P_{\mathbf{x}}[n] = E\left[\mathbf{x}^2[n]\right] \quad (47b)$$

5 Wide-Sense Stationary (WSS) Process

Definition:

$$E[\mathbf{x}(t)] = E[\mathbf{x}(0)] = \mu_{\mathbf{x}} = \text{const} \quad (48a)$$
$$R_{\mathbf{x}}(t_1,t_2) = R_{\mathbf{x}}(\tau = |t_2 - t_1|), \quad \forall t_1,t_2 \quad (48b)$$
$$E[\mathbf{x}[n]] = E[\mathbf{x}[0]] = \mu_{\mathbf{x}} = \text{const} \quad (48c)$$
$$R_{\mathbf{x}}[n_1,n_2] = R_{\mathbf{x}}(k = |n_2 - n_1|), \quad \forall n_1,n_2 \quad (48d)$$

- Auto-correlation

$$R_{\mathbf{x}}(\tau) = E[\mathbf{x}(t)\mathbf{x}(t+\tau)] \quad (49a)$$
$$R_{\mathbf{x}}[k] = E\left[\mathbf{x}[n]\mathbf{x}[n+k]\right] \quad (49b)$$

  Properties:

$$R_{\mathbf{x}}(-\tau) = R_{\mathbf{x}}(\tau) \quad (50a)$$
$$R_{\mathbf{x}}(0) = E[\left|\mathbf{x}(0)\right|^2] = E[\left|\mathbf{x}(t)\right|^2] \quad (50b)$$
$$\mathrm{Var}\left[\mathbf{x}(t)\right] = C_{\mathbf{x}}(0) = \sigma_{\mathbf{x}}^2 \quad (50c)$$
$$R_{\mathbf{x}}(0) \geqslant \left|R_{\mathbf{x}}(\tau)\right| \quad (50d)$$

- Auto-covariance

$$C_{\mathbf{x}}(\tau) = R_{\mathbf{x}}(\tau) - \mu_{\mathbf{x}}^2 \quad (51a)$$
$$C_{\mathbf{x}}[k] = R_{\mathbf{x}}[k] - \mu_{\mathbf{x}}^2 \quad (51b)$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(\tau) = \frac{C_{\mathbf{x}}(\tau)}{C_{\mathbf{x}}(0)} \quad (52a)$$
$$\rho_{\mathbf{x}}[k] = \frac{C_{\mathbf{x}}[k]}{C_{\mathbf{x}}[0]} \quad (52b)$$

5.1 Power Spectral Density (PSD)

$$S_{\mathbf{x}}(F) = \mathcal{F}\left\{R_{\mathbf{x}}(\tau)\right\} = \qquad -\infty \leq F \leq \infty$$
$$= \int_{-\infty}^{\infty} R_{\mathbf{x}}(\tau)\exp\left(-j2\pi F\tau\right)d\tau \quad (53a)$$
$$R_{\mathbf{x}}(\tau) = \mathcal{F}^{-1}\left\{S_{\mathbf{x}}(F)\right\} =$$
$$= \int_{-\infty}^{\infty} S_{\mathbf{x}}(f)\exp\left(j2\pi F\tau\right)dF \quad (53b)$$
$$S_{\mathbf{x}}(f) = \mathrm{DTFT}\left\{R_{\mathbf{x}}[k]\right\} = \sum_{k=-\infty}^{\infty} R_{\mathbf{x}}[k]e^{-j2\pi fk} \quad (53c)$$

Properties:

$$S_{\mathbf{x}}(F) = S_{\mathbf{x}}(-F) \quad (54a)$$
$$S_{\mathbf{x}}(F) \geqslant 0,\ \forall F \quad (54b)$$
$$S_{\mathbf{x}}(F) \in \mathbb{R} \quad (54c)$$
$$S_{\mathbf{x}}(f) = S_{\mathbf{x}}(-f) \quad (54d)$$
$$S_{\mathbf{x}}(f) \geqslant 0,\ \forall f \quad (54e)$$
$$S_{\mathbf{x}}(f) \in \mathbb{R} \quad (54f)$$
$$S_{\mathbf{x}}(f) = S_{\mathbf{x}}(f+1) \quad (54g)$$

Average power

$$P_{\mathbf{x}} = E\left[\mathbf{x}^2(t)\right] = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(F)dF \quad (55a)$$

$$P_{\mathbf{x}} = E\left[\mathbf{x}^2[n]\right] = R_{\mathbf{x}}[0] = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{\mathbf{x}}(f)df \quad (55b)$$

5.2 White Noise & White Gaussian Noise (WGN) Process

White noise process is SSS (WSS) process that is characterized by

$$R_{\mathbf{n}}(\tau) = \sigma^2\delta(\tau) \quad (56a)$$

$$S_{\mathbf{n}}(F) = \sigma^2 \quad \forall F \quad (56b)$$

For WGN process, $\mathbf{n}(t) \sim N(0, \sigma^2)$,

$$R_{\mathbf{n}}(\tau) = \frac{N_0}{2}\delta(\tau) \quad (57a)$$

$$S_{\mathbf{n}}(F) = \frac{N_0}{2} \quad \forall F \quad (57b)$$

5.3 Relation Between Covariance Matrix & Auto-covariance

Given WSS process $\mathbf{x}(t)$, the corresponding correlation matrix of $\mathbf{X} = [\mathbf{x}(t_1), \ldots, \mathbf{x}(t_N)]^T$ is given by

$$R_{\mathbf{X}} = E\left[\mathbf{X}\mathbf{X}^T\right] \quad (58)$$

$$R_{\mathbf{X}}(i,j) = E\left[X_i X_j\right] = R_{\mathbf{x}}\left(|t_i - t_j|\right) \quad (59)$$

6 Cross-Signal

- Cross-correlation

$$R_{\mathbf{xy}}(t_1, t_2) = E[\mathbf{x}(t_1)\mathbf{y}(t_2)] \quad (60)$$

- Cross-covariance

$$C_{\mathbf{xy}}(t_1, t_2) = R_{\mathbf{xy}}(t_1, t_2) - E[\mathbf{x}(t_1)]E[\mathbf{y}(t_2)] \quad (61)$$

- Correlation Coefficient

$$\rho_{\mathbf{xy}}(t_1, t_2) = \frac{C_{\mathbf{xy}}(t_1, t_2)}{\sqrt{C_{\mathbf{x}}(t_1, t_1)C_{\mathbf{y}}(t_2, t_2)}} \quad (62)$$

6.1 WSS Cross-signal

- $\mathbf{x}(t), \mathbf{y}(t)$ are jointly WSS, if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ each of them is WSS and

$$R_{\mathbf{xy}}(\tau) = E[\mathbf{x}(t)\mathbf{y}(t+\tau)] \quad (63)$$

- When $\mathbf{x}(t)$ and $\mathbf{y}(t+\tau)$ are *uncorrelated jointly WSS*, $C_{\mathbf{xy}}(\tau) = 0$.

  Properties

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{yx}}(-\tau) \quad (64a)$$

$$\left|R_{\mathbf{xy}}(\tau)\right| \leqslant \sqrt{R_{\mathbf{x}}(0)R_{\mathbf{y}}(0)} \quad (64b)$$

$$\left|R_{\mathbf{xy}}(\tau)\right| \leqslant \frac{1}{2}\left[R_{\mathbf{x}}(0) + R_{\mathbf{y}}(0)\right] \quad (64c)$$

- Cross-covariance

$$C_{\mathbf{xy}}(\tau) = R_{\mathbf{xy}}(\tau) - \mu_{\mathbf{x}}\mu_{\mathbf{y}} \quad (65)$$

- Cross-PSD

$$S_{\mathbf{xy}}(f) = \mathcal{F}\left\{R_{\mathbf{xy}}(\tau)\right\} \quad (66)$$

  Properties

$$S_{\mathbf{xy}}(f) = S_{\mathbf{yx}}(-f) = S^*_{\mathbf{xy}}(-f) \quad (67)$$

  Correlation coefficient

$$\rho_{\mathbf{xy}}(\tau) = \frac{C_{\mathbf{xy}}(\tau)}{C_{\mathbf{xy}}(0)} \quad (68)$$

- Coherence

$$\gamma_{\mathbf{xy}}(f) = \frac{S_{\mathbf{xy}}(f)}{\sqrt{S_{\mathbf{x}}(f)S_{\mathbf{y}}(f)}} \quad (69)$$

7 LTI and WSS Random Process

Output of LTI system with impulse response $h(t)$ and random process $x(t)$,

$$y(t) = x(t) * h(t) \quad (70)$$

Average

$$m_{\mathbf{y}} = m_{\mathbf{x}} \int_{-\infty}^{\infty} h(s)ds = m_{\mathbf{x}}H(F=0) \quad (71)$$

Cross-correlation & cross-covariance:

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) \tag{72a}$$
$$C_{\mathbf{xy}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) \tag{72b}$$
$$R_{\mathbf{yx}}(\tau) = R_{\mathbf{x}}(\tau) * h(-\tau) \tag{72c}$$
$$C_{\mathbf{yx}}(\tau) = C_{\mathbf{x}}(\tau) * h(-\tau) \tag{72d}$$
$$R_{\mathbf{y}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \tag{72e}$$
$$C_{\mathbf{y}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \tag{72f}$$

Power-Spectral Density (PSD) & Cross-PSD: Given frequency response

$H(F) = \mathcal{F}\left\{h(\tau)\right\},\ H^*(F) = \mathcal{F}\left\{h(-\tau)\right\}$

$$S_{\mathbf{xy}}(F) = S_{\mathbf{x}}(F)\, H(F) \tag{73a}$$
$$S_{\mathbf{yx}}(F) = S_{\mathbf{x}}(F)\, H^*(F) \tag{73b}$$
$$S_{\mathbf{y}}(F) = S_{\mathbf{x}}(F)\left|H(F)\right|^2 \tag{73c}$$

Power of the process:

$$P_y = R_{\mathbf{y}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(F)\left|H(F)\right|^2 dF \tag{74}$$

Same process passes two different systems

$$R_{\mathbf{yz}}(\tau) = R_{\mathbf{x}}(\tau) * h_1(-\tau) * h_2(\tau) \tag{75}$$
$$S_{\mathbf{yz}}(F) = S_{\mathbf{x}}(F) H_1^*(F) H_2(F) \tag{76}$$

7.1 Discrete-Time

Auto-correlation

$$H(z) = \mathscr{Z}\left\{h[n]\right\} = \frac{B(z)}{A(z)}$$
$$\mathscr{Z}\left\{h[n] * h[-n]\right\} = \frac{B(z)B(z^{-1})}{A(z)A(z^{-1})}$$
$$S_{\mathbf{x}}(z) = \mathscr{Z}\left\{R_{\mathbf{x}}[n]\right\}$$
$$h[k] * h[-k] = \sum_m h[m]h[m+k]$$

PSD

$$S_{\mathbf{xy}}(z) = S_{\mathbf{x}}(z)H(z) \tag{77a}$$
$$S_{\mathbf{yx}}(z) = S_{\mathbf{x}}(z)H(z^{-1}) \tag{77b}$$
$$S_{\mathbf{y}}(z) = S_{\mathbf{x}}(z)H(z)H(z^{-1}) \tag{77c}$$

Two different systems

$$R_{\mathbf{yz}}[k] = R_{\mathbf{x}}[k] * h_1[-k] * h_2[k] \tag{78a}$$
$$S_{\mathbf{yz}}(f) = S_{\mathbf{x}}(f)H_1^*(f)H_2(f) \tag{78b}$$
$$S_{\mathbf{yz}}(z) = S_{\mathbf{x}}(z)H_1(1/z)H_2(z) \tag{78c}$$

Power of the process:

$$P_x = R_{\mathbf{x}}[0] = \int_{-1/2}^{1/2} S_{\mathbf{x}}(f)\, df \tag{79a}$$
$$P_y = R_{\mathbf{y}}[0] = \int_{-1/2}^{1/2} S_{\mathbf{x}}(f)\left|H(f)\right|^2 df \tag{79b}$$

Average

$$\mu_{\mathbf{y}} = \mu_{\mathbf{x}} \sum_m h[m] \tag{80}$$

For white Gaussian noise input

$$\mathrm{Var}\left[\mathbf{y}[n]\right] = \mathrm{Var}\left[\mathbf{x}[n]\right] \sum_m h^2[m] \tag{81}$$

7.2 Gaussian Process

A Gaussian process $\mathbf{x}(t)$ a random process that for $\forall k > 0$ and for all times $t_1, \ldots, t_k$, the set of random variable $\mathbf{x}(t_1), \ldots, \mathbf{x}(t_k)$ is jointly Gaussian.

Properties:

- WSS Gaussian process is SSS.
- Gaussian process $\mathbf{x}(t)$ that passes through LTI system, $\mathbf{y}(t) = h(t) * \mathbf{x}(t)$, is also Gaussian process that may be described by the change of expectation and auto-correlation,

$$E[\mathbf{y}(t)] = E[\mathbf{x}(t)] \int_{-\infty}^{\infty} h(s)ds \tag{82a}$$
$$= E[\mathbf{x}(t)]H(0), \quad H(F) = \mathscr{F}\left\{h(t)\right\}$$
$$C_{\mathbf{y}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \tag{82b}$$

- The resulting autocorrelation may be used for producing the correspondent covariance matrix $C_{\mathbf{Y}}$ of a multivariate Gaussian $\mathbf{Y} = [\mathbf{y}(t_1), \ldots, \mathbf{y}(t_N)]^T$

7.3 Linear Prediction

Given N samples of process $\mathbf{x}[n]$, and predictor

$$\hat{\mathbf{x}}[n+1] = \sum_{i=1}^{N} a_i \mathbf{x}[n-i+1], \tag{83}$$

the mean-square error is given by

$$\begin{aligned} mse &= E\left[\left(\mathbf{x}[n+1] - \hat{\mathbf{x}}[n+1]\right)^2\right] \\ &= E\left[\left(\mathbf{x}[n+1] - a_0\mathbf{x}[n] - a_1\mathbf{x}[n-1] - \ldots - a_N\mathbf{x}[n-N]\right)^2\right] \end{aligned} \tag{84}$$

and the values of a_i are given by a solution of

$$\begin{bmatrix} R_{\mathbf{x}}[0] & R_{\mathbf{x}}[1] & \cdots & R_{\mathbf{x}}[N-1] \\ R_{\mathbf{x}}[1] & R_{\mathbf{x}}[0] & \cdots & R_{\mathbf{x}}[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ R_{\mathbf{x}}[N-1] & R_{\mathbf{x}}[N-2] & \cdots & R_{\mathbf{x}}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} R_{\mathbf{x}}[1] \\ R_{\mathbf{x}}[2] \\ \vdots \\ R_{\mathbf{x}}[N] \end{bmatrix} \tag{85}$$

and the resulting minimum MSE is

$$mse_{min} = R_{\mathbf{x}}[0] - \sum_{i=1}^{N} a_i R_{\mathbf{x}}[i] \tag{86}$$

7.4 Match Filter

The goal of filter $h(t)$ is to provide maximum SNR at time $t = t_0$ for *deterministic* signal $x(t)$ and noise $n(t)$.

$$H(f) = \alpha \frac{X^*(f)}{S_n(f)} e^{-j2\pi f t_0} \tag{87}$$

$$y(t) = \frac{1}{\alpha} R_v(t - t_0) \tag{88}$$

For white noise, $n(t)$, with $S_N(f) = N_0/2$, the filter is given by

$$H_{\text{mf}}(f) = X^*(f)e^{-j2\pi f t_0} \longleftrightarrow h_{\text{mf}}(t) = x(t_0 - t) \tag{89}$$

and the resulting maximum SNR is given by

$$SNR_{\max} = \frac{2}{N_0} \int_{-\infty}^{\infty} \left|X(f)\right|^2 df = \frac{2E_x}{N_0} \tag{90}$$

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8 Different Supplementary Formulas

8.1 Derivatives

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\frac{d}{dx} \exp\left[f(x)\right] = \exp\left[f(x)\right] \frac{d}{dx} f(x)$$

8.2 Integrals

8.2.1 Indefinite

$$\int x^n dx = \frac{1}{n+1} x^{n+1},\ n \neq -1$$

$$\int \exp(ax) dx = \frac{1}{a} \exp(ax)$$

$$\int x \exp(ax) dx = \exp(ax) \left[\frac{x}{a} - \frac{1}{a^2}\right]$$

$$\int x^2 \exp(ax) dx = \exp(ax) \left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right]$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\int \cos(ax + b) dx = \frac{\sin(ax + b)}{a}$$

8.2.2 Definite

$$\int_0^{\infty} \exp\left(-a^2 x^2\right) dx = \frac{\sqrt{\pi}}{2a}$$

$$\int_0^{\infty} x^2 \exp\left(-a^2 x^2\right) dx = \frac{\sqrt{\pi}}{4a^3}$$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x - a) dx = f(a)$$

8.3 Fourier Transform

8.3.1 Properties

$$\frac{d^n}{dt^n}g(t) \overset{\mathscr{F}}{\Longleftrightarrow} (j2\pi F)^n G(F)$$
$$g(-t) \overset{\mathscr{F}}{\Longleftrightarrow} G^*(F)$$
$$g(t-t_0) \overset{\mathscr{F}}{\Longleftrightarrow} G(F)e^{-j2\pi F t_0}$$
$$g(t)e^{j2\pi f_0 t} \overset{\mathscr{F}}{\Longleftrightarrow} G(F-F_0)$$

8.3.2 Transform pairs

$$u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2}\left(\frac{1}{j\pi F} + \delta(F)\right)$$
$$\exp(-at)u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{a+j2\pi F}$$
$$t\exp(-at)u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{(a+j2\pi F)^2}$$
$$\exp\big(-a|t|\big) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{2a}{a^2+4\pi^2F^2}$$
$$\exp\left(-at^2\right) \overset{\mathscr{F}}{\Longleftrightarrow} \sqrt{\frac{\pi}{a}}\exp\left(-\frac{(\pi F)^2}{a}\right)$$
$$\cos(2\pi f_a t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2}\Big[\delta(F-F_a)+\delta(F+F_a)\Big]$$
$$\sin(2\pi f_a t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2j}\Big[\delta(F-F_a)-\delta(F+F_a)\Big]$$
$$u(t+a)-u(t-a) \overset{\mathscr{F}}{\Longleftrightarrow} \mathrm{sinc}(2\pi F a) \qquad \text{pulse in time}$$
$$\mathrm{sinc}(2\pi F a) \overset{\mathscr{F}}{\Longleftrightarrow} u(F+a)-u(F-a)$$

8.4 Convolution

$$x(t)*y(t) = \int_{-\infty}^{\infty} f(s)g(t-s)ds$$
$$x(t)*y(t) \overset{\mathscr{F}}{\Longleftrightarrow} X(F)Y(F)$$
$$\delta(t)*y(t) = y(t)$$

8.5 Trigonometry

$$\sin^2(\alpha) = \frac{1}{2}\big(1-\cos(2\alpha)\big)$$
$$\cos^2(\alpha) = \frac{1}{2}\big(1+\cos(2\alpha)\big)$$
$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$
$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$
$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\Big[\sin(\alpha-\beta)+\sin\big((\alpha+\beta)\big)\Big]$$

8.6 Matrices

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
$$\det[\mathbf{A}] = ad-bc$$
$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

9 Discrete-Time

Series sum

$$\sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r}$$
$$\sum_{n=N_1}^{N_2-1} r^n = \frac{r^{N_1}-r^{N_2}}{1-r} \quad N_1 \le N_2$$
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad |r|<1$$
$$\sum_{n=N_1}^{\infty} r^n = \frac{1}{1-r^{N_1}} \quad |r|<1$$
$$\sum_{n=0}^{\infty} nr^n = \frac{r}{(1-r)^2} \quad |r|<1$$

9.1 Z-transforms

$$X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$

Signal	Z transform	ROC
$\delta[n]$	1	$\mathbb{C}$
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	$\mathbb{C} - \{0\}$ if $m > 0$, $\mathbb{C} - \{\infty\}$ if $m < 0$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > a$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < a$

Property	Discrete Signal	Z transform	ROC
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(z) + a_2 X_2(z)$	includes $R_1 \cap R_2$
Time shift	$x[n - n_0]$	$z^{-n_0} X(z)$	R
Frequency scaling	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$\|z_0\|R$
Time reversal	$x[-n]$	$X(z^{-1})$	R^{-1} if $m < 0$
Convolution	$(x_1 * x_2)[n]$	$X_1(z) X_2(z)$	$R_1 \cap R_2$ (or possibly more)
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{X(z)}{1-z^{-1}}$	$R \cap \{\|z\| > 1\}$

9.2 DTFT

$$X(f) = X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\omega}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{jn\omega} d\omega.$$